Published by Institute of Physics Publishing for SISSA/ISAS

Received: *January 17, 2004*
Accepted: *February 9, 2004*

Crosscaps in Gepner models and type-IIA orientifolds

Suresh Govindarajan

Department of Physics, Indian Institute of Technology
Madras, Chennai 600 036 India
Email: suresh@chaos.iitm.ernet.in

Jaydeep Majumder

Department of Physics and Astronomy, Rutgers University
Piscataway, NJ 08855-0849 U.S.A.
Email: jaydeep@physics.rutgers.edu

Abstract: As a first step to a detailed study of orientifolds of Gepner models associated with Calabi-Yau manifolds, we construct crosscap states associated with anti-holomorphic involutions (with fixed points) of Calabi-Yau manifolds. We argue that these orientifolds are dual to M-theory compactifications on (singular) seven-manifolds with G_2 holonomy. Using the spacetime picture as well as the M-theory dual, we discuss aspects of the orientifold that should be obtained in the Gepner model. This is illustrated for the case of the quintic.

Keywords: D-branes, Superstring Vacua, Conformal Field Models in String Theory.

Contents

1. **Introduction and summary** **2**

2. **Geometric analysis** **5**
   2.1 The quintic hypersurface in $\mathbb{P}^4$ 6
   2.2 M-theory on $CY_3 \times S^1$: massless spectrum 7
   2.3 M-theory on Joyce manifolds: the massless spectrum 8

3. **Orientifold projection in the CFT** **10**
   3.1 $N = 2$ preliminaries 10
   3.2 Vertex operators for various fields 11
      3.2.1 Supersymmetry Charges 11
      3.2.2 Gravity multiplet and the universal hypermultiplet 12
      3.2.3 Vertex operators for the hypermultiplets 13
      3.2.4 Vertex operators for the vector multiplet 13
   3.3 The orientifold projection 13
      3.3.1 Hypermultiplets 13
      3.3.2 Gravity and Vector multiplets 14

4. **Basics of unoriented strings** **15**
   4.1 Techniques of open descendants 16
   4.2 Order N simple currents and corresponding open descendants 17

5. **Crosscap states for Gepner models** **18**
   5.1 Boundary and crosscap states in the minimal model 18
   5.2 Discrete symmetries in the minimal model 19
   5.3 Discrete automorphism group of $(k = 1)^3$ and $(k = 3)^5$ Gepner models 20
   5.4 The spacetime sector 21
   5.5 The Gepner model 21
      5.5.1 The β_r-projection 22
      5.5.2 The β_0-projection 22

6. **A toy model: $(k = 1)^3$ Gepner model** **23**
   6.1 A-type crosscap states of $(k = 1)^3$ model 24
      6.1.1 The crosscaps 24
   6.2 Spectra, character formulae and spectral flow invariant orbits of $k = 1$ minimal model 25
   6.3 KB amplitude 25

7. KB amplitudes in $(k=3)^5$ Gepner model **28**
7.1 (2,2) SCFT spectrum 28
7.2 Spectra and spectral flow invariant orbits of $k=3$ minimal model 29
7.3 KB amplitudes 30

8. Conclusion **32**

A. Notation for boundary and crosscap states in RCFT **33**

B. Constraints on open string amplitudes **33**

C. Formulae for simple currents **34**

D. $\mathcal{N}=2$ minimal models **35**
D.1 S and P matrices for $\mathrm{U}(1)_k$ 37
D.2 S and P matrices for $\mathrm{SU}(2)_k$ 37
D.3 Y-tensor elements for $\mathrm{SU}(2)_k$ 38
D.4 S and P matrices for the minimal model 38
D.5 S and P matrices for $\mathrm{SO}(2d)_1$ 38
D.6 $\mathcal{N}=2$ characters 38

E. Character formulae of $k=1$ minimal model **40**

F. Character formulae and spectral flow invariant orbits of $k=3$ minimal model **41**

---

1. Introduction and summary

M-theory compactifications involving *compact* seven-dimensional manifolds of G_2 holonomy are of interest because they lead to compactifications with minimal i.e., $\mathcal{N}=1$ supersymmetry in four dimensions [1]–[9](see ref. [10] for a review and a list of references). Other options to obtain minimal supersymmetry include compactifications of the heterotic string on CY threefolds and F-theory on CY fourfolds. In some cases, all these might be related to each other by some form of duality.

It is also useful to consider M-theory to be the strong coupling limit of the type-IIA string theory. A large class of G_2 manifolds, the *Joyce manifolds*, have been constructed by Joyce using orbifold methods [11, 12]. One particular class, called "barely G_2 manifold", is obtained by orbifolding the manifold $\mathrm{CY}_3 \otimes S^1$. We shall consider the CY_3 to be at the Gepner point of its moduli space. By generalising an argument due to Kachru and McGreevy [13], we will argue that barely G_2 Joyce manifolds appear as the strong coupling limit of certain CY_3 orientifolds at the Gepner point. We will study these orientifolds using several different approaches. These fall into two classes: the spacetime approach and the worldsheet approach. Within the latter, we take the boundary state approach. Construction of orientifolds *viz.* crosscaps states and computing Klein bottle amplitudes out of those crosscaps in Gepner model will be the main theme of this paper. For models of phenomenological interest, e.g., type-IIA orientifolds on $T^6/(\mathbb{Z}_2 \otimes \mathbb{Z}_2)$ and their lift to M-theory on G_2 manifolds, see refs. [14, 15, 16].

Orientifolds [17]–[20] of compact Calabi-Yau manifolds has been a subject of recent interest [21]–[24].[1] Orientifolds of WZW CFT and coset CFTs had been investigated in refs. [26]–[29].[2] The most thorough investigation of orientifolds of generic rational conformal field theory, e.g., bosonic parafermions, $\mathcal{N}=2$ minimal models and Calabi-Yau from the CFT point of view had been done in refs. [31, 32]. For earlier attempts on orientifolds of Gepner models, mainly type-I theory on Gepner models, see refs. [33, 34]. D-branes in coset CFTs had been studied in ref. [35]; Study of D-branes in Gepner models was initiated in [36] and carried out thoroughly for the case of quintic in ref. [37].

Our method will be example oriented because we believe the general procedure for constructing orientifolds in rational conformal field theory(RCFT) can be best understood if we focus on particular examples. We found that Gepner models constructed out of level k_r minimal model with $k_r =$ even for some r, is quite intricate as compared to the $k_r =$ odd counterpart. This is also reflected in the geometric limit as the anti-holomorphic involutions and their fixed point sets are much richer. For simplicity, in this paper we mainly deal with Gepner models with $k_r =$ odd for all r, reserving the detailed investigation on $k_r =$ even models for future work. Moreover, within $k_r =$ odd Gepner models we pick those with only one Kähler modulus; the canonical choice for such CY_3 is the quintic.

We start by considering M-theory on a barely G_2 manifold $\mathrm{CY}_3 \otimes S^1/\sigma_0 \cdot I_1$ where σ_0 is an antiholomorphic involution[3] of CY_3 and I_1 is the inversion of the circle S^1. For CY_3 we consider the Fermat's quintic hypersurface and its corresponding Gepner model $(k=3)^5$. As we go to the $g_s \to 0$ limit, the circle S^1 shrinks in size and we are left with type-IIA theory on the orientifold [13, 39, 40] $\mathrm{CY}_3/\Omega \cdot \sigma(-1)^{F_L}$, where Ω is the worldsheet parity and F_L denote left-moving spacetime fermion number.[4] Suppose, somebody asks at this point the following question: what are the crosscap states in this theory? Answering this question was our main motivation and subject of this paper.

We now outline the organization of our paper. We start in section 2 with the geometric approach first by working out the massless spectra of M-theory on barely G_2 manifold. In section 3, we check this spectra by working in the worldsheet approach. Here we work out the action of the orientifold projection operator on each massless state in $\mathcal{N}=2$, $c=9$ superconformal algebra and work out the massless fields in the projected theory. It agrees with the analysis of section 2 and also with the rules in ref. [41], given years ago by working on the geometric side.

[1] For type-II theories on toroidal orientifolds in pre-D-brane era, see refs. [25].

[2] See the very recent paper ref. [30] which relate TFT and RCFT on unoriented worldsheet.

[3] The image of σ_0 in type IIA theory will be denoted as σ.

[4] When we consider the Gepner model of such CY_3, the orientifold group will also contain the discrete symmetry group of the Gepner model.

Section 4 contain the basic set up for treating unoriented strings in RCFT with and without simple currents. It mainly summarizes the foundations laid in refs. [42, 43]. The reader is requested to go through this section first, since it contains all the major formulae and forms the backbone for subsequent sections. Other informations relevant to the section 4 can be found in appendices A, B and C.

Sections 6 and 7 form the heart of the paper. As a warm-up and to prepare the groundwork for the orientifold of the quintic model, we discuss the type-IIA orientifold on $(k=1)^3$ Gepner model in section 6. The orientifold group can be enlarged by incorporating the discrete symmetries of the Gepner model. This can be found in section 5.3. The most important formula for the P-matrix is given in appendix D.4. The explicit expressions for the crosscaps in *internal* CFT are given in section 6.1.1. To get a physical insight for the Klein bottle amplitude of this model, we first discuss the spectral-flow invariant orbits for $(k=1)^3$ Gepner model. In section 6.3, we compute the Klein bottle amplitude for this model by using the crosscaps of section 6.1.1 and express the answer in terms of those spectral-flow invariant orbits. The *master formula* for the Klein bottle amplitude can be found in eq. (6.9) in section 6.1.1. Though these formulae for crosscaps and Klein bottle amplitudes are derived for $(k=1)^3$ Gepner model, it is general enough in that it can be applied to any Gepner model with all levels k_r being *odd*; in particular, it can be applied to the quintic in a straightforward way. As expected, we get the Klein bottle amplitude for this model to be proportional to the *massless* orbit of the model. It was quite satisfying and show that the abstract RCFT construction is on the right track. This warm-up example gave us enough confidence to apply this technology directly to the quintic Gepner model. The explicit formulae for the characters of $k=1$ minimal model can be found in appendix E.

In section 7, we discuss our main goal, i.e. the crosscaps and Klein bottle amplitudes in quintic. It is easy now to write down the crosscaps using the method of section 6.1.1 and is given in eq. (7.7). Before computing the Klein bottle amplitude in $(k=3)^5$ model, we write down the spectral flow invariant orbits of this model. Though we searched hard in the literature, we think this is for the first time the spectral flow invariant orbits are being worked out. There are many of them but we mention the most important of them in section 7.2. As a cross check, we compare it with the characters of $c=9$ algebra — the worldsheet algebra relevant for manifolds of SU(3) holonomy [44, 45]. After discussing the orientifold group of this model, we compute the Klein bottle amplitude in section 7.3. The answer is given in eq. (7.9); as expected it only involves the massless graviton and self-conjugate matter orbits. For further details, see the relevant section. We gather the formulae for the characters and *string functions* for $k=3$ minimal model in the appendix F.

We conclude in section 8. In appendix D we gather various important formulae for $\mathcal{N}=2$ minimal model.

As we were finishing this project, we received a preprint [78], which has overlaps with our section 5. They proposed a crosscap similar to our eq. (5.19) in section 5, though we proposed it much earlier [77] and this issue had been clearly addressed in our PASCOS '03 conference report ref. [79].

JHEP02(2004)026

2. Geometric analysis

We shall briefly discuss the Joyce's construction of seven-dimensional manifolds with barely G_2-holonomy. Consider the seven-dimensional orbifold X given by the $\mathbb{Z}_2$ action on the manifold $M\times S^1$, where M is a Calabi-Yau manifold admitting an anti-holomorphic involution σ and the $\mathbb{Z}_2$ is given by an inversion I_1 of the S^1 and the anti-holomorphic involution σ. Thus,

$$X = \frac{(M \times S^1)}{\sigma \cdot I_1}\,. \tag{2.1}$$

The anti-holomorphic involution σ is an isometry of the Calabi-Yau manifold; it acts on its Kähler form J and the holomorphic 3-form $\Omega^{(3)}$ as:

$$\sigma(J) = -J\,, \qquad \sigma(\Omega^{(3)}) = e^{i\,\theta}\,\Omega^{(3)}\,, \tag{2.2}$$

where θ is a real phase. This action of σ on X and the inversion I_1 on the circle S^1 preserves the 3-form

$$\phi = J \wedge d\,x + + \mathrm{Re}\,(e^{i\theta/2}\,\Omega^{(3)}) \tag{2.3}$$

on M and equips it with a G_2 structure; hence the projection $\sigma \cdot I_1$ preserves $\mathcal{N} = 1$ supersymmetry in $D = 4$.

One distinguishes two cases: (i) the involution σ has no fixed points and (ii) the involution σ has a submanifold $\Sigma \in M$ as its fixed point. For case (i), X is a non-singular manifold with G_2 holonomy while for case (ii) X is a singular manifold with G_2 holonomy. Joyce shows that if $b_1(\Sigma) > 0$, the singularity can be blown up to obtain a non-singular manifold with G_2-holonomy. When $b_1(\Sigma) = 0$, the manifold has a non-smoothable singularity.

For purposes of compactifications of M-theory on manifolds with G_2 holonomy, singularities are a virtue — they provide us with examples of $\mathcal{N} = 1$ supersymmetric compactifications in four dimensions with non-abelian gauge symmetry as well as chiral fermions, both of phenomenological interest [5, 7].

Anti-holomorphic involutions also make an appearance in a different context. They provide us with a large class of special lagrangian (sL) submanifolds of Calabi-Yau threefolds. Thus, Σ can be chosen to be sL submanifold of the CY threefold. In such cases, we will argue, extending an argument due to Kachru and McGreevy [13], that the type-IIA dual of the M-theory compactification on X is a type-IIA orientifold of M, with the S^1 playing the role of the eleven-dimensional circle:

$$\mathrm{M-theory\ on\ X} \overset{\mathrm{dual}}{\longleftrightarrow} \mathrm{Orientifold\ of\ type\ IIA\ on\ M}\,.$$

The appearance of non-abelian gauge symmetry is easily seen in the orientifold due to the addition of D6-branes wrapping Σ and extending in the non-compact spacetime in order to cancel the RR-tadpoles due to the presence of orientifold 6-planes. Suppose $\Sigma = L_1 + L_2$, where L_1 and L_2 are two sL submanifolds which preserve the same supersymmetry and intersect each other at a point. Then, following ref. [46], one expects the appearance of chiral fermions at the points of intersection.

We need to map the inversion of the eleven dimensional circle to a suitable orientifold action in order to obtain the correct type-IIA dual for the M-theory compactification. Let us first consider the type-IIA string in flat spacetime. Inversion of an odd number of coordinates is not a symmetry — it can however be made a symmetry by including the simultaneous action of worldsheet parity and possibly $(-)^{F_L}$. The precise choice depends on the number of coordinate inversions. From the analysis of Sen [39, 56], one finds that the correct choices are given by

$$\mathbb{R}^{9-p}\mathcal{I}_{9-p} \cdot \Omega \cdot g\,, \tag{2.4}$$

where $\mathcal{I}_{9-p}$ reverses the sign of all coordinates on $\mathbb{R}^{9-p}$, Ω is the worldsheet parity operation and

$$g = \begin{cases} 1 & \text{for } (9-p) = 0,1 \mod 4 \\ (-)^{F_L} & \text{for } (9-p) = 2,3 \mod 4\,. \end{cases} \tag{2.5}$$

The origin is the fixed point of the orientifold action and is the location of an orientifold p-plane. The RR-charge of the orientifold plane can be (locally) cancelled by the addition of $32/2^{9-p}$ Dp-branes in the convention that one adds 32 D9-branes to obtain type-I theory as a type-IIB orientifold. The enhanced gauge symmetry is $\mathrm{SO}(32/2^{9-p})$ for an SO-type orientifold plane (O^+ in Witten's convention).

2.1 The quintic hypersurface in $\mathbb{P}^4$

The most studied example of a compact CY threefold is given by the quintic hypersurface in $\mathbb{P}^4$. The Fermat quintic M_{FQ} is given by the hypersurface given by equation

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0\,, \tag{2.6}$$

where z_i are homogeneous coordinates of $\mathbb{P}^4$. The fixed point of the anti-holomorphic involution

$$\sigma : \quad z_i \to \bar{z}_i\,, \qquad i = 1, \ldots, 5\,, \tag{2.7}$$

is an $\mathbb{RP}^3$, which is a sL submanifold of the Fermat quintic [64]. This submanifold is also the base of SYZ T^3-fibration of the Fermat quintic [65]. The anti-holomorphic involution corresponds to the inversion of all three circles of the T^3-fibre. From equation ((2.4)), the anti-holomorphic involution can be made a symmetry of type-IIA string by including worldsheet parity with $(-)^{F_L}$. Thus, the duality with M-theory can be made precise for this example

$$\mathrm{M-theory\ on\ } M_{\mathrm{FQ}} \times S^1/\sigma \cdot I_1 \overset{\text{dual}}{\longleftrightarrow} \text{type IIA on } \mathrm{M_{FQ}}/\sigma \cdot \Omega \cdot (-)^{\mathrm{F_L}} \tag{2.8}$$

Tadpole considerations from the flat case suggest that IIA orientifold will need the addition of four D6-branes wrapping the $\mathbb{RP}^3$ and filling spacetime. We will see that all this will be consistent with our CFT analysis in the next section.

Σ is actually one in a family of $5^4 = 625$ sL submanifolds of the Fermat quintic, all of whom are $\mathbb{RP}^3$'s. In fact the involution given in eq. (2.7) is a special case of eq. (2.2) with $\theta = 0$. The most general anti-holomorphic involution with a non-vanishing θ are given by the following:

$$z_i \to \omega^{n_i} \bar{z}_i\,, \tag{2.9}$$

JHEP02(2004)026

where $\omega = e^{2\pi i/5}$ and the identification under a shift of all n_i by one is understood. This is a trivial scaling of all homogeneous coordinates. This freedom can be used to set $n_5 = 0$. All these sL manifolds are further divided by $\sum_i n_i$ mod 5 — this is related to the phase associated with the sL condition.

In ref. [59, 60], all possible anti-holomorphic involution for the embedding projective space $\mathbb{CP}^N$ had been classified. They are of two types. If $z^1,\ z^2,\cdots,\ z^{N+1}$ denote the homogenous coordinates of $\mathbb{CP}^N$, then these anti-holomorphic involutions are:

$$\begin{aligned} A\ &:\ (z^1, z^2, \ldots, z^{N+1})\ \rightarrow\ (\bar{z}^1, \bar{z}^2, \ldots, \bar{z}^{N+1}) \\ B\ &:\ (z^1, z^2, \ldots, z^N, z^{N+1})\ \rightarrow\ (-\bar{z}^2, \bar{z}^1, \ldots, -\bar{z}^{N+1}, \bar{z}^N)\,. \end{aligned} \tag{2.10}$$

Here B is defined only for *odd* N and acts *freely* while A has a fixed set(for $z^i = \bar{z}^i$). In our examples we mostly deal with A-type involutions. Further, if the embedded Calabi-Yau surface is given as

$$\sum_{i=1}^{N+1} z_i^{n_i} = 0\,,$$

then one can add another discrete symmetry

$$z^i\ \rightarrow\ \zeta_i\, z^i\,, \qquad i = 1\,, \ldots,\ N+1\,, \tag{2.11}$$

to either of the involutions A or B; here ζ_i is n_i-th root of unity. Here $\{\zeta_i\,;\ i = 1\,,\cdots\,,\ N{+}1\}$ generate a subgroup of the automorphism group of the Calabi-Yau hypersurface or the associated Landau-Ginzburg superpotential or of the associated Gepner model. We shall give explicit examples of this dicrete symmetry groups in sections 6 and 7 in the context of $(k = 1)^3$ and $(k = 3)^5$ Gepner models. Together with the antiholomorphic involutions A and worldsheet parity Ω, we shall build the full orientifold group in these models.

2.2 M-theory on $CY_3 \times S^1$: massless spectrum

The massless spectrum can be obtained from compactifying eleven-dimensional supergravity on a CY_3 (as in Witten) and then further dimensionally reducing on a circle. Alternately, one can compactify M-theory on a circle to obtain the type-IIA string and then compactify the IIA string on the CY_3, M. We will pursue the second method. Of course, both methods give the same spectrum.

The compactification of M-theory on a circle gives rise to the type-IIA string whose bosonic spectrum is

$$\begin{array}{rlll} \text{NSNS sector} & G_{\hat\mu\hat\nu} & B_{\hat\mu\hat\nu} \sim C_{\hat\mu\hat\nu\theta} & \Phi \sim G_{\theta\theta} \\ \text{RR sector} & A^{(1)}_{\hat\mu} \sim G_{\hat\mu\theta} & A^{(3)}_{\hat\mu\hat\nu\hat\rho} \sim C_{\hat\mu\hat\nu\hat\rho} & \end{array} \tag{2.12}$$

where we have used hatted Greek indices to denote ten-dimensional vectors and symbolically indicated the eleven-dimensional origin of the various fields using θ to indicate the index for the the S^1 and C for the three-form gauge field.

JHEP02(2004)026

On further compactification on the CY3 M with $h^{1,1}(M)$ Kähler moduli and $h^{2,1}(M)$ complex moduli, the NSNS fields decompose as

$$\begin{aligned}
G_{\hat{\mu}\hat{\nu}} &\longrightarrow \begin{cases} g_{\mu\nu} \\ g_{ij} = z^A \ \mu_{A\bar{i}\bar{j}} \\ g_{i\bar{j}} = v^k \ \omega_k \end{cases} \\
B_{\hat{\mu}\hat{\nu}} &\longrightarrow B_{\mu\nu} \text{ and } b^k \ \omega_{ki\bar{j}} \\
\Phi &\longrightarrow \Phi
\end{aligned} \tag{2.13}$$

where μ_A are the $h^{1,2}(M)$ Beltrami differentials that parametrise deformations of complex structure of M and ω_k are a basis for $H^{1,1}(M)$.

In the RR sector, the one-form gauge field provides a gauge field (the graviphoton) A_μ in four dimensions. The RR three-form $A^{(3)}$ can be decomposed as

$$A^{(3)} = \xi^A \alpha_A + \tilde{\xi}_B \beta^B + C^{(1)\ k} \omega_k \,, \tag{2.14}$$

(α_A, β^B) are a real basis for $H^3(M)$, $A, B = 1, \ldots, h^{2,1}(M)$ with $\alpha_A \wedge \beta^B = \delta_A^B$ and $\alpha_A \wedge \alpha_B = \beta^A \wedge \beta^B = 0$. The holomorphic three-form is taken to be $\alpha_0 + i\beta^0$.

The compactification of the type-II string on a Calabi-Yau threefold gives rise to $\mathcal{N} = 2$ supergravity in $d = 4$ as its low-energy limit. Thus, one can organise the bosonic fields into multiplets of $\mathcal{N} = 2$ supergravity. The full massless spectrum is

1. *The gravity multiplet:* the graviton, $g_{\mu\nu}$; the graviphoton, A_μ and two gravitini.

2. *The universal hypermultiplet:* four real scalars: the dilaton Φ, b (after dualising $B_{\mu\nu}$), ξ^0, $\tilde{\xi}_0$.

3. $h^{2,1}(M)$ *Hypermultiplets:* the complex scalars z^A and the real scalars ξ^A, $\tilde{\xi}_A$ and a Dirac fermion.

4. $h^{1,1}(M)$ *Vector multiplets:* the gauge fields $C_\mu^{(1)\ k}$, complex scalars $t^k = b^k + iv^k$ and a Dirac fermion.

2.3 M-theory on Joyce manifolds: the massless spectrum

As we discussed earlier, Joyce manifolds are manifolds with G_2 holonomy that are obtained as an $\mathbb{Z}_2$ orbifold of CY3 $\times S^1$. The $\mathbb{Z}_2$ is the combination of an anti-holomorphic involution of the CY3 and inversion of the circle.

The orbifold projection breaks $\mathcal{N} = 2$ down to $\mathcal{N} = 1$. It is therefore useful to decompose $\mathcal{N} = 2$ multiplets into $\mathcal{N} = 1$ multiplets. A hypermultiplet breaks into two chiral multiplets, an $\mathcal{N} = 2$ vector multiplet decomposes into an $\mathcal{N} = 1$ vector multiplet and a chiral multiplet. Finally, the $\mathcal{N} = 2$ gravity multiplet decomposes into the $\mathcal{N} = 1$ supergravity multiplet and a $\mathcal{N} = 1$ vector multiplet, which we will call the graviphoton mutiplet.

The orbifold projection is as follows:

1. *The gravity multiplet:* the graviphoton and its supersymmetric partner which is one of the gravitini get projected out leaving behind a $\mathcal{N}=1$ supergravity multiplet.

2. *The universal hypermultiplet:* b is projected out while the dilaton is projected in. One linear combination of the ξ^0 and $\tilde{\xi}^0$ is invariant under the anti-holomorphic involution. Thus, one is left with a $\mathcal{N}=1$ chiral multiplet.

3. *Hypermultiplets:* the inversion of the the S^1 does not affect these fields and hence the anti-holomorphic involution alone plays a role in the projection. This involution leaves invariant a 'real' set of fields: of the two real fields that make up z^A, one linear combination is invariant and is projected in. The same story holds for the ξ^A and $\tilde{\xi}^A$. They form $h^{2,1}$ chiral multiplets.

4. *Vector multiplets:* the inversion of the S^1 does not affect the gauge fields $C_\mu^{(1)\ k}$. The projection is determined wholly by whether the $(1,1)$ form ω_k is even or odd under the anti-holomorphic involution. When ω_k are even, the vector multiplet is projected in and otherwise they are projected out. Among the scalars, b^k is odd under the S^1 inversion and thus it is projected in whenever the corresponding $(1,1)$ form ω_k is odd. Let $h_+^{1,1}$ be the number of even $(1,1)$ forms and $h_-^{1,1}$ be the number of odd $(1,1)$ forms. Thus, one has $h_+^{1,1}$ $\mathcal{N}=1$ vector multiplets and $h_-^{1,1}$ chiral multiplets after the orbifold projection.

Let us first consider this case when the oribifold has no fixed points. The massless spectrum is the same as the compactification of M-theory on a smooth G_2 manifold X with Betti numbers [57, 58, 59]

$$b_3(X) = h^{2,1}(M) + h_-^{1,1}(M)\,, \qquad b_2(X) = h_+^{1,1}(M)\,,$$

i.e., there are $b_3(X)$ chiral multiplets and $b_2(X)$ abelian vector multiplets in addition to the supergravity multiplet.

Our main focus will be on the cases when one has fixed points. There are various possibilities if the seven-manifold X is singular. Non-abelian (ADE) vector multiplets with A–D–E-type gauge group G arise in M-theory compactifications at singularities of the form $\mathbb{C}^2/\Gamma_{ADE}$. This is the simplest of all possibilities and our case belongs to this category.[5] Thus, when the singularities are of the form $\Sigma \times \mathbb{C}^2/\Gamma_{ADE}$, we expect extra non-abelian vector multiplets; here Σ is a 3-manifold embedded in X. When $b_1(\Sigma) \neq 0$, we expect to see the appearance of $b_1(\Sigma)$ chiral multiplets which can be understood as the blowup modes and according to Joyce, the singularity can be smoothed out. When $b_1(\Sigma)=0$, the singularity is non-smoothable and the non-abelian vector multiplets remain.

[5]If Σ is smooth and the normal space to Σ is a smoothly varying family of A–D–E singularities, the $(3+1)$-dimensional low energy theory will be a theory with gauge group G, without chiral matter. In this case the dimension of the moduli space of the low energy theory is equal to $b_1(\Sigma)$. To get chiral matter, Σ must be singular or it must pass through worse than orbifold singularities of X. In that case the dimension of the moduli space of low energy theory gets bigger than $b_1(\Sigma)$, since now one has to consider the moduli of complex gauge connection along Σ; see ref. [61] for more details. We thank Bobby Acharya for a discussion on this point.

We shall aim to reproduce these results in our analysis of the orientifold which we have proposed as the dual to these M-theory compactifications on these Joyce manifolds.

3. Orientifold projection in the CFT

In this section, we will work out the action of the orientifold group on the vertex operators associated with various fields that appear in the Calabi-Yau compactification of the type-IIA string.

3.1 $N = 2$ preliminaries

The worldsheet has enhanced supersymmetry, $(2, 2)$ with generators $(T_L(z), G^{\pm}_L(z), J_L(z))$ and $(T_R(\bar{z}), G^{\pm}_R(\bar{z}), J_R(\bar{z}))$ for the left- and right-moving sector respectively, each generating an $N = 2$ algebra. Primary states are thus labelled by four numbers (h_L, q_L, h_R, q_R), where h_L (h_R) is the conformal weight and q_L (q_R) is the $\mathrm{U}(1)_L$ $(\mathrm{U}(1)_R)$ charge. The $N = 2$ superconformal algebra has a parameter $a(0 \leq a < 1)$ related to boundary conditions on the fermionic generators or equivalently on the moding of these generators [66, 67]. In the complex plane, the fermionic generators have integer moding $(a = 0)$ in the Ramond sector and half-integer moding $(a = 1/2)$ in the NS sector.

In the NS sector, there are a special class of primaries which satisfy $h = |q|/2$. These are the *chiral* primaries with $h = +q/2$ and *anti-chiral* with $h = -\frac{q}{2}$. In the Ramond sector, the ground states in a unitary theory can be shown to have $h = c/24$. There is a mapping, the *spectral flow*, which relates states in the Ramond sector to the NS sector. In general, the $N = 2$ algebras given by a gets mapped to one with $(a + \eta)$ by means of spectral flow ith spectral parameter η. Under this action, the primary given by (h, q) gets mapped to the primary given by

$$\begin{aligned} h &\to h_\eta = h - \eta\, q + \frac{c}{6}\,\eta^2 \\ q &\to q_\eta = q_L - \frac{c}{3}\,\eta\,. \end{aligned} \tag{3.1}$$

From the above formulae, one can see that under a spectral flow with $\eta = 1/2$, chiral primaries get mapped to Ramond ground states and under spectral flow with $\eta = -1/2$, anti-chiral primaries get mapped to Ramond ground states.

Recall that there are two $N = 2$ algebras, one each from the left- and right- movers. We shall label the spectral parameters η_L and η_R respectively. The operators which generate spectral flow can be written out explicitly on bosonisation. Let

$$J_L(z) = i\sqrt{\frac{c}{3}}\,\partial_z\, H_L(z) \qquad \text{and} \qquad J_R(\bar{z}) = i\sqrt{\frac{c}{3}}\,\partial_{\bar{z}}\, H_R(\bar{z}) \tag{3.2}$$

with the normalization given by

$$H_L(z)\, H_L(w) \sim -\ln(z - w) \qquad \text{and} \qquad H_R(\bar{z})\, H_R(\overline{w}) \sim -\ln(\bar{z} - \overline{w})\,.$$

In terms of the bosons $H_L(z)$ and $H_R(\bar{z})$ the spectral flow operator corresponding to spectral parameters (η_L, η_R) is given by

$$U_{\eta_L,\eta_R} = e^{i\sqrt{\frac{c}{3}}(\eta_L\, H_L + \eta_R\, H_R)}\,. \tag{3.3}$$

Sector	NSNS $\xrightarrow{(\eta_L,\eta_R)}$ RR	Number
Identity	$1 \xrightarrow{(\frac{1}{2},-\frac{1}{2})} \Sigma^0$	1
Identity	$1 \xrightarrow{(-\frac{1}{2},\frac{1}{2})} \Sigma^{0\dagger}$	1
Identity	$1 \xrightarrow{(\frac{1}{2},\frac{1}{2})} \Xi^0$	1
Identity	$1 \xrightarrow{(-\frac{1}{2},-\frac{1}{2})} \Xi^{0\dagger}$	1
(c,a)	$\Lambda^j \xrightarrow{(\frac{1}{2},-\frac{1}{2})} \Sigma^j$	$j=1,\ldots,h_{1,1}$
(a,c)	$\Lambda^{j\dagger} \xrightarrow{(-\frac{1}{2},\frac{1}{2})} \Sigma^{j\dagger}$	$j=1,\ldots,h_{1,1}$
(c,c)	$\Pi^A \xrightarrow{(\frac{1}{2},\frac{1}{2})} \Xi^A$	$A=1,\ldots,h_{2,1}$
(a,a)	$\Pi^{A\dagger} \xrightarrow{(-\frac{1}{2},-\frac{1}{2})} \Xi^{A\dagger}$	$A=1,\ldots,h_{2,1}$

Table 1: States and their relation under spectral flow.

We are interested in compactifications of type-IIA on Calabi-Yau threefolds. The Calabi-Yau sector has $c = 9$ and the massless bosonic states arise from two sectors: NSNS and RR. Spectral flow relates states in these sectors and is closely related to supersymmetry in spacetime. We will now tabulate the relevant states. The massless states in the NSNS sector arise from the (c, c), (a, a) primaries [there are $h_{2,1}$ of these] and (a, c) and (c, a) primaries [there are $h_{1,1}$ of these] with $h = 1/2$. We used the obvious notation: c for chiral primaries and a for anti-chiral primaries. Finally, there is the identity operator which is both chiral and anti-chiral and has $h = 0$. In table 1, we set the notation for the operators corresponding to these massless excitations(see for instance, refs. [18, 69])

3.2 Vertex operators for various fields

The vertex operators for the various fields will be needed to study the action of the orientifold group and implement in the projection. It is natural to work in the $(-1, -1)$ picture for the NSNS vertex operators, $(-1/2, -1/2)$ picture for RR vertex operators and the $(-1/2, 0)$ and $(0, -1/2)$ pictures for the generators of spacetime supersymmetry. The vertex operators include pieces from the spacetime sector as well as the ghost sector. The index μ is a vector index of $SO(3, 1)$ and $\alpha(\dot\alpha)$ are indices for Weyl spinors with positive(negative) chirality(see ref. [71] for notation). The (free) fields that make up the spacetime sector are X^μ, ψ^μ_L and ψ^μ_R.

3.2.1 Supersymmetry Charges

The type-IIA compactification has $\mathcal{N} = 2$ supersymmetry in four dimensions. The two supersymmetry charges arise from the R-NS and the NS-R sectors. charges. Let us label them by

$$Q^1 = \begin{pmatrix} Q^1_\alpha \\ Q^1_{\dot\alpha} \end{pmatrix}; \qquad Q^2 = \begin{pmatrix} Q^2_\beta \\ Q^2_{\dot\beta} \end{pmatrix}, \tag{3.4}$$

where

$$
\begin{aligned}
Q^1_\alpha &= \oint d\,z\,e^{-\varphi_L/2}\,(S_L)_\alpha\,\exp\left[\frac{i\sqrt{3}}{2}H_L\right](z)\\
Q_{1\,\dot\alpha} &= \oint d\,z\,e^{-\varphi_L/2}\,(\overline{S}_L)^{\dot\alpha}\,\exp\left[-\frac{i\sqrt{3}}{2}H_L\right](z)\\
Q^2_\beta &= \oint d\,\bar z\,e^{-\varphi_R/2}\,(S_R)_\beta\,\exp\left[-\frac{i\sqrt{3}}{2}H_R\right](\bar z)\\
Q^{2\,\dot\beta} &= \oint d\,\bar z\,e^{-\varphi_R/2}\,(\overline{S}_R)^{\dot\beta}\,\exp\left[\frac{i\sqrt{3}}{2}H_R\right](\bar z)
\end{aligned} \tag{3.5}
$$

with S_α and $S_{\dot\alpha}$ as given above are the spin fields of SO(3,1) obtained by bosonising the fermions ψ^μ in the spacetime sector. The above choice reflect the spinorial content of the Ramond ground state in ten-dimensions. For the IIA string, the chiralities are opposite in the left and right sectors. Let us choose them to be $\mathbf{8_s}$ (i.e., positive ten-dimensional chirality) for the left-movers and $\mathbf{8_c}$ for the right movers. From the Calabi-Yau sector, the chirality of the Ramond ground states is reflected in sign of U(1) charge. Since ten-dimensional chirality must be given by the product of the four-dimensional and internal(CY) chirality, one has: $\mathbf{8_s}\to(\alpha,+)\oplus(\dot\alpha,-)$ and $\mathbf{8_c}\to(\alpha,-)\oplus(\dot\alpha,+)$. The above choices for the supersymmetry charges reflect this.

3.2.2 Gravity multiplet and the universal hypermultiplet

The vertex operators for the NSNS fields, i.e., the graviton, B-field and the dilaton in the $(-1,-1)$ picture are:

$$
V^{(-1,-1)}(k,\zeta)=\zeta_{\mu\nu}\,e^{-\varphi_L-\varphi_R}\,\psi^\mu_L(z)\,\psi^\nu_R(\bar z)\,e^{ik\cdot X}(z,\bar z) \tag{3.6}
$$

where $k^2=k^\mu\,\zeta_{\mu\nu}=k^\nu\,\zeta_{\mu\nu}=0$. The *symmetric traceless* part of $\zeta_{\nu\mu}$ gives the graviton vertex operator, the *antisymmetric* part, the vertex operator for the B-field and the *trace*, the dilaton vertex operator.

The states that come from the RR sector are the graviphoton: (ε_μ be the corresponding polarization vector)

$$
\begin{aligned}
V^{(-\frac12,-\frac12)}_{\text{graviphoton}}(k,\varepsilon) &= k_{[\nu}\,\varepsilon_{\mu]}\,e^{-(\varphi_L+\varphi_R)/2}\times\\
&\times\Big[(S_L)_\alpha\,\epsilon^{\alpha\gamma}(\sigma^{[\mu\nu]})_\gamma^{\ \beta}\,(S_R)_\beta\,\Sigma_0(z,\bar z)+\\
&+(\overline{S}_L)^{\dot\alpha}\,\epsilon_{\dot\alpha\dot\gamma}(\overline{\sigma}^{[\mu\nu]})^{\dot\gamma}_{\ \dot\beta}\,(\overline{S}_R)^{\dot\beta}\,\Sigma^{0\dagger}(z,\bar z)\Big]e^{i\,k\cdot X}(z,\bar z)\,.
\end{aligned} \tag{3.7}
$$

The RR scalars ξ^0 and ξ_0 that form the universal hypermultiplet (with the dualised B-field and the dilaton) are given by the vertex operators:

$$
V^{(-\frac12,-\frac12)}_{\xi^0+i\xi_0}(k)=k_\mu\,e^{-(\varphi_L+\varphi_R)/2}\Big[(S_L)_\alpha\,\epsilon^{\alpha\gamma}(\sigma^\mu)_{\gamma\dot\beta}\,(\overline{S}_R)^{\dot\beta}\,\Xi^0(z,\bar z)\Big]\,e^{i\,k\cdot X}(z,\bar z)\,. \tag{3.8}
$$

Note the appearance of the four-momentum k in the RR vertex operators — these reflect the fact that they couple to field strengths of p-form gauge fields.

3.2.3 Vertex operators for the hypermultiplets

The vertex operators for the scalars z^A that arise from the NSNS sector are given by

$$V_{(z^A)}^{(-1,-1)}(k) = e^{-(\varphi_L+\varphi_R)}\,\Pi^A(z,\bar{z})\,e^{i\,k\cdot X}(z,\bar{z})\,. \tag{3.9}$$

The scalars ξ^A and ξ_A from the RR sector arise from

$$V_{(\xi^A+i\xi_A)}^{(-\frac{1}{2},-\frac{1}{2})}(k) = k_\mu\,e^{-(\varphi_L+\varphi_R)/2}\Big[(S_L)_\alpha\,\epsilon^{\alpha\gamma}(\sigma^\mu)_{\gamma\dot\beta}\,(\overline{S}_R)^{\dot\beta}\,\Xi^A(z,\bar{z})\Big]\,e^{i\,k\cdot X}(z,\bar{z})\,. \tag{3.10}$$

3.2.4 Vertex operators for the vector multiplet

The vertex operator for such scalars t^j in $(-1,\,-1)$-picture is given by:

$$V_{t^j}^{(-1,-1)}(k) = e^{-(\varphi_L+\varphi_R)}\,\Lambda^j(z,\bar{z})\,e^{i\,k\cdot X}(z,\bar{z}) \tag{3.11}$$

with the one for $\bar{t}^j$ obtained by the replacement $\Lambda^j \to \Lambda^{j\,\dagger}$. The vertex operator of the vector field in the multiplet is given by

$$\begin{aligned}V_{C^j_\mu}^{(-\frac{1}{2},-\frac{1}{2})}(k,\bar\epsilon) \;=\; & k_{[\,\nu}\,\bar\varepsilon_{\mu\,]}\,e^{-(\varphi_L+\varphi_R)/2}\times \\ & \times\Big[(S_L)_\alpha\,\epsilon^{\alpha\gamma}(\sigma^{[\mu\nu]})_\gamma^{\ \beta}\,(S_R)_\beta\,\Sigma_j(z,\bar{z}) + \\ & + (\overline{S}_L)^{\dot\alpha}\,\epsilon_{\dot\alpha\dot\gamma}(\overline{\sigma}^{[\mu\nu]})^{\dot\gamma}_{\ \dot\beta}\,(\overline{S}_R)^{\dot\beta}\,\Sigma^{j\dagger}(z,\bar{z})\Big]e^{i\,k\cdot X}(z,\bar{z})\,. \end{aligned} \tag{3.12}$$

3.3 The orientifold projection

Our discussion so far holds for general anti-holomorphic involutions. We will now specialise to the case where the anti-holomorphic involution σ is complex conjugation. A naive guess for the action of σ on an NSNS field $\Phi^{q_L\,q_R}_{h\,\bar h}$ is($q_L,\,q_R>0$):[6]

$$\sigma\;:\;\Phi^{q_L\,q_R}_{h\,\bar h}\;\to\;\Phi^{-q_L\,-q_R}_{h\,\bar h}\,. \tag{3.13}$$

This can be seen to be consistent with the fact that in the LG model corresponding to the Gepner model, all chiral superfields get mapped to anti-chiral superfields. Of course, as we have seen, σ in itself is not a symmetry but the combination $\mathcal{O}\equiv\sigma\cdot\Omega$ is. The action of Ω interchanges left and right movers. Thus, using the above rule, it is easy to see that (c,c) states get mapped to (a,a) states while $(a,c)((c,a))$ states get mapped to $(a,c)((c,a))$ states.

3.3.1 Hypermultiplets

It is thus not hard to see that for massless NSNS modes that arise from the (c,c) and (a,a) states, that one linear combination of the two scalars that make up z^A survives the orientifold projection. In the RR sector, one has (for $A=0,1,\ldots,h_{1,2}$)

$$(S_L)_\alpha(\overline{S}_R)^{\dot\beta}\;\Xi^A\;\stackrel{(-)^{F_L}\cdot\mathcal{O}}{\longleftrightarrow}\;(S_L)^{\dot\beta}(S_R)_\alpha\;\Xi^{A\dagger}$$

and thus one linear combination of the scalars ξ^A and ξ_A survives the projection. They combine to form a $\mathcal{N}=1$ chiral multiplet.

[6]The case when $q_L=q_R=0$ is more complicated. For the case of single minimal models, for low k, see the discussion in ref. [68].

3.3.2 Gravity and Vector multiplets

Let us choose to write the identity operator as Λ^0. The action of $\mathcal{O}$ on the fields Λ^j (choosing a diagonal basis) is (no summation over j below)

$$\mathcal{O}\,\Lambda^j\,\mathcal{O}^{-1} = \nu_j\,\Lambda^j\,, \tag{3.14}$$

with $\nu_0 = +1$ and $\nu_j = \pm 1$ for $j = 1, \ldots, h_{1,1}$. The spectral flow operator $U_{\frac{1}{2},-\frac{1}{2}}$ maps these operators to the Ramond ground states Σ^j. $\mathcal{O}$ has the following action on $U_{\frac{1}{2},-\frac{1}{2}}$:

$$\mathcal{O}\,U_{\frac{1}{2},-\frac{1}{2}}\,\mathcal{O}^{-1} = \eta\,U_{\frac{1}{2},-\frac{1}{2}}\,, \tag{3.15}$$

where we have included a possible sign. This implies that

$$\mathcal{O}\,\Sigma^j\,\mathcal{O}^{-1} = \eta\,\nu_j\,\Sigma^j\,. \tag{3.16}$$

Finally, we need the action of $\mathcal{O}$ on the fermions ψ_L^μ and ψ_R^μ:

$$\mathcal{O}\,\psi_L^\mu\,\psi_R^\nu\,\mathcal{O}^{-1} = \psi_L^\nu\,\psi_R^\mu\,. \tag{3.17}$$

Using eq. (3.17), one can see that the symmetric part of the first vertex operator is projected in implying that the graviton and dilaton are projected in. Now, Using equation (3.14), we can see that when $\nu_j = +1$, the scalars that come from the second vertex operator above, is projected in.

Sector	NSNS	RR
Graviton	Symmetric part	–
$\nu_j = +1$	scalars	–
$\nu_j = -1$	–	vector

Table 2: Summary of the projection in gravity and vector multiplets.

In order to work out the orientifold projection, we need to consider the action of $\mathcal{O}$ and $(-)^{F_L}$ on the spinfields.

$$\mathcal{O}(S_L)_\alpha\,(S_R)_\beta\;\mathcal{O}^\dagger = (S_R)_\alpha(S_L)_\beta = -(S_L)_\beta\,(S_R)_\alpha \tag{3.18}$$

where the minus sign comes from the cocycles (see for instance, [18, 72]) or equivalently from the fact that the spin fields are spacetime fermions. Under $(-)^{F_L}$, one has

$$(S_R)_\alpha(S_L)_\beta \xrightarrow{(-)^{F_L}} -(S_L)_\alpha\,(S_R)_\beta\,. \tag{3.19}$$

Thus, under the combined action of $\mathcal{O}$ and $(-)^{F_L}$, one obtains

$$(S_R)_\alpha\,(S_L)_\beta\,\Sigma^j \xrightarrow{\mathcal{O}\cdot(-)^{F_L}} \eta\,\nu_j\,(S_L)_\beta\,(S_R)_\alpha\,\Sigma^j\,. \tag{3.20}$$

Thus, for $\eta\nu_j = +1$, the symmetric part is projected in. One can show that $\epsilon^{\alpha\gamma}(\sigma^{[\mu\nu]})_\gamma^{\ \beta}$ is a symmetric matrix(see appendix A, [71]) and hence the gauge field is projected in. In particular, we would like to see that the graviphoton has to be projected out, i.e., we need $\eta\nu_0 = \eta = -1$. Thus, we need $\eta = -1$. Once this choice is made, we see that, for $\nu_j = -1$, the vector multiplet is projected in. Here is a summary of states projected in given in table 2. Comparing with the M-theory analysis, we see that $\nu_j = -1$ corresponds to the Kähler class be even. These results are consistent with the geometric analysis of ref. [41].

4. Basics of unoriented strings

In this section we summarize the main results of [42] and [43] for constructing RCFTs on unoriented worldsheets with and without boundary. At the level of CFT's, an orientifold introduces surfaces with crosscaps, i.e. unoriented string sectors.[7] We start with a known modular invariant torus partition function for oriented closed string sector

$$\mathcal{T} = \sum_{i\,,j} \chi_i\, Z_{i\,j}\, \overline{\chi_j}\,, \tag{4.1}$$

with

$$\chi_i(\tau) = \mathrm{Tr}_{\mathcal{H}_i}\, q^{L_0 - c/24}\,, \qquad q = e^{2\pi\, i\, \tau} \tag{4.2}$$

where $Z_{i\,j}$ is a symmetric modular invariant matrix and χ_i is the character of the representation i; here $\mathcal{H}_i$ is the Hilbert space for the i-th irreducible representation of the algbera.

Klein bottle: Suppose the orientifold group is[8] $\mathcal{O} = \Omega{\cdot}\sigma$, where σ is an anti-holomorphic involution of the CY_3; in CFT language it corresponds to *simple currents* of the underying worldsheet algebra. Since crosscap interchanges chiral with anti-chiral fields of the 2d RCFT, therefore,[9]

$$\mathcal{O}\,:\,\varphi_{i\,i^c} \to \epsilon_i\, \varphi_{i^c\,i}\,; \qquad \mathcal{O}^2 = \mathbf{1} \quad \epsilon_i = \pm 1\,, \tag{4.3}$$

where $\{\varphi_{i\,i^c}\}$ denote the set of primary fields of the RCFT, labelled by their conformal weights $(h_i,\, \bar{h}_i)$. The *direct channel* Klein bottle amplitude is defined as follows:

$$\begin{aligned}\mathcal{K}_{\substack{\mathrm{NSNS}\\ \mathrm{RR}}}(q) &= \frac{1}{2}\,\mathrm{Tr}_{\substack{\mathrm{NSNS}\\ \mathrm{RR}}}\left[\Omega\cdot\sigma\,(1+(-1)^F\ )q^{H_{cl}}\right]\\ &= \frac{1}{2}\sum_i \mathcal{K}^i\, \chi_i^{\substack{(\mathrm{NSNS})\\ (\mathrm{RR})}}(q)\,,\end{aligned} \tag{4.4}$$

where F is the worldsheet fermion number operator, $\mathcal{K}^i$ are integers and $H_{cl} = \frac{1}{2}(L_0 + \overline{L}_0 - \frac{c}{12})$ is the closed string hamiltonian. The condition that the closed string sector $\frac{1}{2}(\mathcal{T}+\mathcal{K})$ has positive, integral multiplicities for all states requires that [43]

$$\mathcal{K}_i = \epsilon_i\, Z_{i\,i}\,, \tag{4.5}$$

where we define $\epsilon_i = 0$ when $Z_{i\,i} = 0$.[10] In the *transverse channel* Klein bottle amplitude is interpreted as the propagation of closed strings between two crosscap states:

$$\widetilde{\mathcal{K}}(\tilde{q}) = \left\langle C \middle|\, \tilde{q}^{H_{cl}} \,\middle| C \right\rangle = \sum_i \Gamma^i\, \Gamma^i\, \chi_i(\tilde{q})\,, \tag{4.6}$$

[7]It may or may not introduce (unoriented) open string sectors, i.e. D-branes. It very much depend on the nature of the orientifold.

[8]$\mathcal{O}$ may or may not contain $(-1)^{F_L}$ factor.

[9]For notation and conventions, see appendix A.

[10]Eq. (4.5) is not all. It is consistent (under the operator product expansion) iff $\epsilon_i\,\epsilon_j\,\epsilon_k\,\mathcal{N}_{i\,j}^{\ \,k} \geq 0$, where $\mathcal{N}_{i\,j}^{\ \,k}$ are the fusion matrix elements, given by the Verlinde formula [49]: $\mathcal{N}_{i\,j}^{\ \,k} = \sum_l \frac{S_{i\,l}\,S_{j\,l}\,S^\dagger_{k\,l}}{S_{0\,l}}$.

JHEP02(2004)026

where in the last step we have used eq. (A.4) and $\tilde{q} = e^{-2\pi i/\tau}$. Since eqs. (4.4) and (4.6) are related by modular transformation matrix S, we have the following (consistency) condition

$$\mathcal{K}_i = \sum_j S_i{}^j \, \Gamma_j \, \Gamma_j \, . \tag{4.7}$$

Möbius strip: generically, the unoriented closed string theory is inconsistent due to the presence of massless tadpoles. One can get rid of them by introducing open string sectors, i.e., D-branes. At the worldsheet level, the relevant 1-loop open string amplitudes which contain these open string sectors are the annulus and the Möbius strip. The Möbius strip amplitudes are defined in terms of real characters:

$$\widehat{\chi}_i = e^{i\,\pi\,(h_i - c/24)}\, \chi_i\!\left(\frac{i\,\tau+1}{2}\right) = (\sqrt{T})^{-1}\, \chi_i\!\left(\frac{i\,\tau+1}{2}\right). \tag{4.8}$$

The modular transformation matrix connecting the direct and transverse channels for the Möbius amplitudes is[11] given by [50]:

$$P = \sqrt{T}\, S\, T^2\, S\, \sqrt{T}\,, \tag{4.10}$$

The direct and transverse channel Möbius strip amplitudes are

$$\begin{aligned} \text{Direct} \ &: \ \mathcal{M}_{\substack{\mathrm{NS}_a \\ \mathrm{R}_a}}(\tau) = \mathrm{Tr}_{\substack{\mathrm{NS}_a \\ \mathrm{R}_a}} \left(\Omega\, h\, q^{L_0 - c/24} \right) = \sum_i \mathcal{M}^i_{\substack{\mathrm{NS}_a \\ \mathrm{R}_a}}\, \widehat{\chi}_i^{\substack{\mathrm{NS}_a \\ \mathrm{R}_a}}(\tau)\,, \\ \text{Transverse} \ &: \ \widetilde{\mathcal{M}}_{\substack{\mathrm{NS}_a \\ \mathrm{R}_a}} = \left\langle C \middle| \, \tilde{q}^{H_{cl}} \, \middle| B, \mathrm{NSNS}_a \mathrm{RR}_a \right\rangle = \sum_i \Gamma^i\, B^i_{\substack{\mathrm{NSNS}_a \\ \mathrm{RR}_a}}\, \widehat{\chi}_i^{\substack{\mathrm{NS}_a \\ \mathrm{R}_a}}(\tilde{q})\,, \end{aligned} \tag{4.11}$$

where $\mathcal{M}^i_{\substack{\mathrm{NS}_a \\ \mathrm{R}_a}}$ represents twists of open string spectra in NS- and R-sectors respectively; these are non-negative integers and a specifies the boundary condition on the boundary. Also $|B\,;\, {}^{\mathrm{NSNS}_a}_{\mathrm{RR}_a}\rangle$ denotes either NSNS- or RR- boundary state for a D-brane satisfying a set of boundary conditions a. As in case of Klein bottle, the total Möbius strip amplitude is obtained by asumming the contribution of NS- and R-sectors of open string channel. The channel transformation matrix P relates $\mathcal{M}_a(\tau)$ and $\widetilde{\mathcal{M}_a}$ as follows:

$$\mathcal{M}^i_{\ a} = \sum_j \Gamma^j\, B^j_{\ a}\, P_j^{\ i}\,, \quad \forall \quad i,\, a\,. \tag{4.12}$$

4.1 Techniques of open descendants

Given a modular invariant, $Z_{i\,j^c}$ and a consistent Klein bottle projection, $\mathcal{K}_i$ in the closed string sector of a RCFT, subject to the constraints given in the appendix B, the answer to the correct open string spectra or in other word, the correct annulus and Möbius coefficients, $\mathcal{A}^{\,i}_{\ a\,b}$, $\mathcal{M}^{\,i}_{\ a}$ has been given in ref. [43]. Henceforth, these solutions will be referred to as PSS solutions. We just quote their results here only for Klein bottles and Möbius strips. There are two types of solutions; the first one being the well-known Cardy type [42].

[11] Like S, P is a unitary and symmetric matrix and satisfies

$$P^2 = \mathcal{C}\,, \qquad P^* = \mathcal{C}\, P = P\, \mathcal{C}\,. \tag{4.9}$$

Cardy-type solutions: we start with a $\mathcal{C}$-diagonal modular invariant, *viz.* , $Z_{i\,j} = \delta_{i\,j^c}$. The values of Klein bottle and Möbius strip coefficients for the $\mathcal{C}$-diagonal case can be expressed in terms of an **integer-valued tensor**(either +ve or −ve) [43]:[12]

$$Y_{i\,j}^{\;k} = \sum_m \frac{S_{i\,m}\,P_{j\,m}\,P^*_{k\,m}}{S_{0\,m}}\,. \tag{4.13}$$

In terms of $Y_{i\,j}^{\;k}$, the Möbius, Klein Bottle amplitudes, boundary and crosscap coefficients are:

$$\textbf{Möbius} \;:\; \mathcal{M}^{\,i}_{\;a} = Y_{a\,0}^{\;i} \tag{4.14}$$

$$\textbf{Klein Bottle} \;:\; \mathcal{K}^{\,i} = Y^{\,i}_{0\,0} \tag{4.15}$$

$$\textbf{Crosscap coefficients} \;:\; \Gamma_i = \frac{P_{i\,0}}{\sqrt{S_{i\,0}}} \quad \Rightarrow \quad |C\rangle = \sum_i \frac{P_{i\,0}}{\sqrt{S_{i\,0}}}\,|C_i\rangle \tag{4.16}$$

Eq. (4.16) is the basic building block for constructing crosscap states in Gepner model and will be used in sections 6 and 7 for constructing crosscaps in $(k=1)^3$ and $(k=3)^5$ Gepner models.

4.2 Order N simple currents and corresponding open descendants

The other solutions to the algebraic constraints of appendix B are given in terms of S, P and Y matrix elements associated to certain primary fields, called *simple currents*[52]. These extra solutions of those constraints in terms of simple currents are relevant for our purpose because the antiholomorphic involutions, generically denoted as σ in the orientifold group is in one-one correspondence with simple currents of the RCFT. These solutions are given in ref. [53].

$$\textbf{Klein Bottle} \;:\; \mathcal{K}^{i}_{[L]} = e^{2\,\pi\,i\,Q_{L^n}(\Phi_{\,i})}\,Y^{\,i}_{0\,0} = Y^{\,i}_{L\,L}\,, \tag{4.17}$$

$$\textbf{Möbius Strip} \;:\; \mathcal{M}^{i}_{[L^n]\,a} = \mathcal{Y}^{i}_{\;a\,L} \qquad \left|\mathcal{M}^{i}_{[L^n]\,a}\right| \le \mathcal{N}_{a\,a}^{\;L^n\otimes\Phi_{\,i}}$$

$$\mathcal{M}^{i}_{[L^n]\,a} = \mathcal{N}_{a\,a}^{\;L^n\otimes\Phi_{\,i}} \quad \text{mod } 2\,, \quad \forall\, n = 0,\,1,\ldots,\mathsf{N}-1 \tag{4.18}$$

$$\textbf{Crosscap coefficients} \;:\; \Gamma_{\,[L^n]\,i} = \frac{P_{L^n\,i}}{\sqrt{S_{0\,i}}}\,, \quad \forall\, n = 0,\,1,\ldots,\mathsf{N}-1 \tag{4.19}$$

$$\Rightarrow \quad |C\,;\,[L^n]\rangle = \sum_i \frac{P_{L^n\,i}}{\sqrt{S_{0\,i}}}\,|C\,;\,i\rangle\,, \tag{4.20}$$

where $Q_{L^n}(\Phi_{\,i})$ is the monodromy charge of the primary field $\Phi_{\,i}$, as given by eq. (C.5). Like eq. (4.16), eq. (4.20) will be the basic building block for constructing crosscap states associated to some particular simple currents in the Gepner model.

[12]Defining the matrix Y_i as $\left(Y_i\right)_j^{\;k} = Y_{i\,j}^{\;k}$, one can check that these matrices Y_i are mutually commuting and satisfy the fusion algebra: $Y_i\,Y_j = \sum_k \mathcal{N}_{i\,j}^{\;k}\,Y_k$.

5. Crosscap states for Gepner models

5.1 Boundary and crosscap states in the minimal model

The A-type boundary and crosscap states (associated with the simple current $(0, M, S)$) in the minimal model at level k are given by a straightforward application of the Cardy and PSS ansatz to be[13]

$$|B : L, M, S\rangle \equiv \sum_{(l,m,s)}^{\mathsf{FR}} \left(S_{(0,0,0)\ (l,m,s)}\right)^{-\frac{1}{2}} S_{(L,M,S)\ (l,m,s)} \, |B : l, m, s\rangle\rangle \tag{5.1}$$

$$|C : M, S\rangle \equiv \sum_{(l,m,s)}^{\mathsf{FR}} \left(S_{(0,0,0)\ (l,m,s)}\right)^{-\frac{1}{2}} P_{(L,M,S)\ (l,m,s)} \, |C : l, m, s\rangle\rangle \,. \tag{5.2}$$

The S and P-matrices for the minimal model case have been discussed in the appendix D.4. Putting in the values of the S and P-matrices, an explicit expression for the the boundary and crosscap states are(with $(L+M+S)$ =even for the boundary states and $(M+S)$ =even for the crosscap states):[14]

$$\begin{aligned}
|B : L, M, S\rangle &\equiv \sqrt{\frac{1}{\sqrt{2}(k+2)}} \sum_{(l,m,s)}^{\mathsf{FR}} \frac{\sin(L,l)_k}{\sqrt{\sin(l,0)_k}} \exp\left(\frac{i\pi Mm}{k+2} - \frac{i\pi Ss}{2}\right) |B : l, m, s\rangle\rangle \\
|C : M, S\rangle &= \sqrt{\frac{2}{k+2}} \sum_{(l,m,s)}^{\mathsf{FR}} \frac{\sigma_{l,m,s}}{\sqrt{\sin(l,0)_k}} \ \exp i\pi \left(\frac{Mm}{2k+4} - \frac{Ss}{4}\right) \ \delta^{(2)}_{s+S} \times \\
&\quad \times \Big[\sigma_{0,M,S} \ \sin\frac{1}{2}(0,l)_k \ \delta^{(2)}_{l+k} \ \delta^{(2)}_{M+m+k} + \\
&\quad + \sigma_{k,M+k+2,S+2}(-)^{\frac{-l+m-s}{2}} \ \cos\frac{1}{2}(0,l)_k \ \delta^{(2)}_{l} \ \delta^{(2)}_{M+m}\Big] |C : l, m, s\rangle\rangle
\end{aligned} \tag{5.3}$$

where $(l, l')_k = \pi(l+1)(l'+1)/(k+2)$ and $\sigma_{l,m,s} \equiv \widehat{\sigma}_{l,m,s} \nu_m^{(k+2)} \nu_s^{(2)}$. ($\widehat{\sigma}_{l,m,s}$ is defined in eq. (D.10) and $\nu_m^{(k)}$ is defined in eq. (D.13) in appendix D.) We have also used the identity

$$\delta^{(2)}_l \sin\frac{1}{2}(k,l)_k = \delta^{(2)}_l (-)^{l/2} \cos\frac{1}{2}(0,l)_k \ .$$

It is important to note that when S is even, the crosscap states consist of Isibashi states exclusively from the NSNS sector and when S is odd, the crosscap states consist of Isibashi states exclusively from the RR sector.

Under the various identifications, one can verify that the boundary and crosscap states transform as:

$$\begin{aligned}
|B : L, M, S\rangle &= |B : L, M + 2k + 4, S\rangle = |B : M, S + 4\rangle = |B : k - L, M + k + 2, S + 2\rangle \\
|C : M, S + 4\rangle &= (-)^S \ |C : M, S\rangle \,.
\end{aligned} \tag{5.4}$$

[13]Since we shall exclusively deal with A-type crosscap states, henceforth we drop the subscript A.

[14]For the choice of $\widehat{\sigma}$ given in eq. (D.10), the crosscap state is identical to the one constructed by Brunner and Hori [32].

The second line above follows on using $\sigma_{l,m+2k+4,s} = \sigma_{l,m,s+4} = \sigma_{l,m,s}$. The Klein bottle amplitudes for the above crosscap states are [32]

$$\langle C:\tilde{M},\tilde{S}|\tilde{q}^{H_{cl}}|C:M,S\rangle = \sigma_{0,\tilde{M},\tilde{S}}\,\sigma_{0,M,S}\,\delta^{(2)}_{M+\tilde{M}}\,\delta^{(2)}_{S+\tilde{S}}\times \\ \times\left[\sum_{l}^{ev}(-)^l\left(\chi^{(\frac{\tilde{S}-S}{2})}_{l,\frac{\tilde{M}-M}{2}}(q)+(-)^S\chi^{(\frac{\tilde{S}-S}{2}+2)}_{l,\frac{\tilde{M}-M}{2}}(q)\right)+\right. \\ \left.+\,\delta^{(2)}_k\left(\chi^{(\frac{\tilde{S}-S+2}{2})}_{\frac{k}{2},\frac{\tilde{M}-M+k+2}{2}}(q)+(-)^S\chi^{(\frac{\tilde{S}-S-2}{2})}_{\frac{k}{2},\frac{\tilde{M}-M+k+2}{2}}(q)\right)\right]. \tag{5.5}$$

It is useful to observe that (for odd k) when $S-\tilde{S}=0 \mod 4$, the characters $NS \equiv \mathrm{Tr}_{NS}[\Omega\, q^{L_0}]$ appear for even S and the characters $R\equiv\mathrm{Tr}_R[\Omega\, q^{L_0}]$ appear for $S-\tilde{S}=2 \mod 4$. For odd S, the characters with the $(-)^F$ insertions appear, $\widetilde{NS}\equiv\mathrm{Tr}_{NS}[\Omega\,(-)^F\, q^{L_0}]$ when $S-\tilde{S}=0 \mod 4$ and $\widetilde{R}\equiv\mathrm{Tr}_R[\Omega\,(-)^F\, q^{L_0}]$ when $S-\tilde{S}=2 \mod 4$.[15] When k is even, the second term contributes in an opposite fashion, i.e., R appears for even S and $S-\tilde{S}=0 \mod 4$ and so on. For the boundary states, a similar role is played by the combination [48]

$$|B:L,M,\pm\rangle \equiv \frac{1}{\sqrt{2}}\left(|B:L,M,S\rangle \pm |B:L,M,S+2\rangle\right), \tag{5.6}$$

One can easily see that $|B:L,M,+\rangle$ involve Ishibashi states from the NSNS sector and $|B:L,M,-\rangle$ involve Ishibashi states from the RR sector. The annulus channel among the overlap of the $|B:L,M,+\rangle$ boundary states involves NS (for even $(L+M)$ and R amplitudes (for odd $(L+M)$) and the $\widetilde{NS}$ and $\widetilde{R}$ characters appear in the annulus channel of the overlap between $|B:L,M,+\rangle$ and the $|B:L',M',-\rangle$. These will be the building blocks for the Gepner model boundary and crosscap states.

5.2 Discrete symmetries in the minimal model

The minimal models have discrete symmetries generated by the simple currents with labels $(0,m,s)$. For odd k, the group is $\mathbb{Z}_{4k+8}$ (generated by the simple current given by the primary with labels $(0,1,1)$) and for even k, the group is $\mathbb{Z}_{2k+2}\times\mathbb{Z}_2$ (generated by the primaries $(0,1,1)$ and $(0,0,2)$). We focus on three generators, which we will call g (corresponds to $(0,2,0)$, h (corresponds to $(0,0,2)$) and f (corresponds to $(0,1,1)$ and generates spectral flow with $\eta=1/2$). Note the identities

$$g\cdot h = f^2\,,\qquad h^2=1 \qquad\text{and}\qquad g^{k+2}=1\,. \tag{5.7}$$

However, these identities may be projectively realised on crosscap states. For instance,

$$h^2\;|C:M,S\rangle = (-)^S\;|C:M,S\rangle\,. \tag{5.8}$$

[15] Here we are somewhat sloppy about our notation; the full orientifold group also includes the generator for antiholomorphic involution, the generators of discrete symmetry group of the Gepner model and $(-1)^{F_L}$ — the presence of the latter being depended on the dimension.

Thus, one will need to add phases into the action of g and h to get their $(k+2)$-th and second powers respectively to equal one on crosscap states.

Under the discrete symmetries of the minimal model generated by g and h, the boundary and crosscap states transform as

$$g \cdot |B : L, M, \pm\rangle = |B : L, M+2, \pm\rangle\,, \qquad h \cdot |B : L, M, \pm\rangle = \pm|B : L, M, \pm\rangle \tag{5.9}$$

$$g \cdot |C : M, S\rangle = |C : M+2, S\rangle\,, \qquad h \cdot |C : M, S\rangle = |C : M, S+2\rangle\,. \tag{5.10}$$

The boundary and crosscap states form orbits of length $(k+2)$ with one exception. When k is even and $L = k/2$, the orbit length is $n = (k+2)/2$.

$$g^n \cdot |B : L, M, \pm\rangle = \pm|B : L, M, \pm\rangle\,.$$

In the context of the Gepner model, this leads to the boundary states which are not minimal and need to be resolved.

5.3 Discrete automorphism group of $(k=1)^3$ and $(k=3)^5$ Gepner models

Once we know the discrete symmetries of each minimal model, we can specify the orientifold group of a Gepner model. This group is model as well as theory(i.e. whether type-IIA or IIB) dependent. As we saw in sections 2 and 2.1, generically for type-IIA compactification down to four spacetime dimensions, this group is given by[16] $G_k = \Omega\cdot\sigma\cdot\mathcal{H}\cdot(-1)^{F_L}$, where $\mathcal{H}$ is some subgroup of the discrete automorphism group of the Gepner models associated to the Calabi-Yau hypersurface. As we shall be dealing with Gepner models with all $k_i = $ odd, from the discussion of the preceding section it is quite clear that the automorphism group is $\mathcal{G} \sim \prod_i \mathbb{Z}_{k_i+2}$ which is generated by $\prod_i g_i$. The group element acts on the NSNS sector stated by multiplying them by $e^{\frac{2\pi i t_i (q_{L,i}+q_{R,i})}{2(k_i+2)}}$ with $t_i \in \mathbb{Z}$.[17] The transformation corresponding to $t_i = (2, 2, \cdots, 2)$ is a trivial one and hence we mod out by the center $\mathbb{Z}_{\tilde{n}} \subset \prod_r \mathbb{Z}_{n_i}$, where $\tilde{n} = \mathrm{lcm}\,(k_i+2)$.[18] Further, we can consider the permutation group which acts by permuting r copies of minimal models amongst themselves. So [62][19]

$$\mathcal{G} = \frac{\prod_i \mathbb{Z}_{n_i}}{\mathbb{Z}_{\tilde{n}}} \otimes S_r = \frac{\prod_i \mathbb{Z}_{k_i+2}}{\mathbb{Z}_{\tilde{n}}} \otimes S_r\,.$$

We do not consider the permutation group and instead consider the subgroup $\mathcal{H} = \prod_r \mathbb{Z}_{k_r+2}/\mathbb{Z}_{\tilde{n}}$. For $(k=1)^3$ model, $\mathcal{H} = \mathbb{Z}_3^3/\mathbb{Z}_3$ and for $(k=3)^5$ model, it is $\mathcal{H} = \mathbb{Z}_5^5/\mathbb{Z}_5$. So the orientifold group for these models are [20]

$$G_{(k=1)^3} = \Omega\cdot\sigma\cdot\frac{\mathbb{Z}_3^3}{\mathbb{Z}_3}\,; \qquad G_{(k=3)^5} = \Omega\cdot\sigma\cdot(-1)^{F_L}\cdot\frac{\mathbb{Z}_5^5}{\mathbb{Z}_5}\,. \tag{5.11}$$

[16]The subscript k in the definition of G reminds us of the level the minimal models used for compactification. The group G depends crucially on it.

[17]For A, D_{odd} and E_6 models, $t_i \in \mathbb{Z}$ and $t_i \in 2\mathbb{Z}$ for D_{even}, E_7 and E_8 models.

[18]If not all k_i are *odd*, $\tilde{n} = \frac{1}{2}\mathrm{lcm}\,(k_i+2)$.

[19]There is an subtlety in defining n_r. It depends on which type of modular invariants are being used to build up the torus parition function of the theory. For A, D_{odd} and E_6 modular invariants, $n_i = k_i+2$. For D_{even}, E_7 and E_8 modular invariants, we have $n_i = (k_i+2)/2$. In our examples of $(k=1)^3$ and $(k=3)^5$ Gepner models, it is always given by (k_i+2).

[20]Since compactification of type IIA theory on $(k=1)^3$ model($c=3$) corresponds to compactification down to 8 spacetime-dimension, the orientifold group does not involve $(-1)^{F_L}$; see eq. (2.5).

5.4 The spacetime sector

We will base our discussion here in the light-cone gauge. So if the non-compact spacetime is D dimensional, then $2d = D-2$. So for the 1^3 Gepner model, $d=3$ and 3^5 Gepner model, $d=1$. Thus, we will focus on the case when d is odd in our examples. The four irreps of $\mathrm{SO}(2d)_1$ are the scalar, vector, spinor and conjugate spinor (O,V,S,C) representations. The weights and charges are $[(0,0),(1/2,\pm1),(d/8,d/2),(d/8,-d/2)]$ respectively. We will represent them by the labels $s_0 = 0,2,1,-1$. Their U(1) charges are given by $(ds_0/2)$ mod 2. The boundary and crosscap states (for SO(2) and SO(6)) are:

$$\begin{aligned}
|B_{\rm st}:S_0\rangle &= \frac{1}{\sqrt{2}}\sum_{s_0}\exp\left(-\frac{id\pi s_0 S_0}{2}\right)|B_{\rm st}:s_0\rangle\rangle \\
|C_{\rm st}:S_0=0,2\rangle &= (-1)^{\frac{(d-1)}{2}\frac{S_0}{2}}\sum_{s_0}\exp\left(-\frac{i\pi s_0 S_0}{4}\right)\times \\
&\quad\times\left[\cos\left(\frac{d\pi}{4}\right)\delta^{(4)}_{S_0-s_0}+\sin\left(\frac{d\pi}{4}\right)\delta^{(4)}_{S_0-s_0+2}\right]|C:s_0\rangle\rangle \\
\Big|C_{\rm st}:S_0=\pm1\Big\rangle &= (-1)^{\frac{(d-1)}{2}\frac{S_0}{2}}\sum_{s_0}\exp\left(-\frac{id\pi}{4}\right)\times \\
&\quad\times\left[\cos\left(\frac{d\pi}{4}\right)\delta^{(4)}_{S_0-s_0}+i\,\sin\left(\frac{d\pi}{4}\right)\delta^{(4)}_{S_0-s_0+2}\right]|C:s_0\rangle\rangle\,. \qquad (5.12)
\end{aligned}$$

We introduce the phase factor in the definition of the crosscap above, to take care of the correct relative sign between the NSNS and RR parts of the KB amplitude for the $(k=1)^3$ Gepner model(here $d=3$). It has no effect for the $(k=3)^5$ model(here $d=1$).

The Klein bottle amplitude that is obtained from the above crosscap states are given by (for odd d)

$$\begin{aligned}
\langle C_{\rm st}:\tilde{S}_0|e^{\frac{-i\pi}{2\tau}H_{cl}}|C_{\rm st}:S_0\rangle &= (-1)^{\frac{(d-1)}{2}\frac{(S_0-\tilde{S}_0)}{2}}\times \\
&\quad\times\frac{1}{2}\Big[\delta^4_{S_0-\tilde{S}_0}\big(\chi^{(0)}_{\rm st}+(-)^{S_0}\chi^{(2)}_{\rm st}\big)(2\tau)+ \qquad (5.13)\\
&\quad+(-)^{S_0}\delta^4_{S_0-\tilde{S}_0+2}\sin\left(\frac{d\pi}{2}\right)\big(\chi^{(1)}_{\rm st}+(-)^{S_0}\chi^{(-1)}_{\rm st}\big)(2\tau)\Big]
\end{aligned}$$

where $\chi^{S_0}_{\rm st}$ are the four $\mathrm{SO}(d)_1$ characters. Note that the spacetime KB amplitudes are quite similar to those that we obtained for odd k_i minimal models. In particular, the discussion after eq. (5.5) holds here as well. For later convenience, we define

$$\chi^{\pm,(NS)}_{\rm st}\equiv(\chi^{(0)}_{\rm st}\pm\chi^{(2)}_{\rm st})\qquad\text{and}\qquad\chi^{\pm,(R)}_{\rm st}\equiv(\chi^{(1)}_{\rm st}\pm\chi^{(-1)}_{\rm st})\,. \qquad (5.14)$$

5.5 The Gepner model

Let us now consider the case of the Gepner model where the internal CFT (corresponding to the Calabi-Yau manifold) is given by a Gepner model, i.e., we consider the tensor product of r minimal models of levels k_i $(i=1,\ldots,r)$. The Gepner projection consists of the following:

1. Tensor NS states with NS states and R states with R states. At the level of partition functions, it must consist of NS characters tensored with other NS characters with similar conditions for the other three types of characters, $\widetilde{NS}$, R and $\widetilde{R}$. We will call this the β_r-projection.

2. Project onto states such that the total U(1) charge is an odd integer. The total U(1) charge has two contributions, one from the spacetime CFT and one from the internal CFT, i.e., the Gepner model. We call this the β_0 projection. This can be done in a two-step process. For the internal CFT, first consider states such that the total U(1) charge is an integer in the NS sector and (for odd d) half-integer in the R-sector. Second, tensor the spacetime part in such a way that the total U(1) charge is an odd integer. So if one has a state obtained from the internal NS sector that has odd(even) U(1) charge, then tensor it with the NS representation of $\mathrm{SO}(2d)_1$ with even charge, i.e., the scalar(vector) representation.

5.5.1 The β_r-projection

As we saw in the discussion of a single minimal model as well as the spacetime sector, the S even crosscap states contain only $NSNS$ Ishibashi states and S odd, RR Ishibashi states. Thus, the first part of the β_r-projection translates into the rule that one must tensor states with even S or odd S. From the Klein bottle amplitude of the individual minimal models, eq. (5.5), for odd k, we see that we should impose something a little more stringent, i.e., require that we tensor all states with identical S. Thus the following tensor product of states (we assume that the level k is odd for all minimal models)

$$|C:\mu,S\rangle \equiv |C_{\rm st}:S\rangle \prod_{i=1}^{r} |C:M_i,S\rangle\,. \tag{5.15}$$

(with $\mu\equiv(M_1,\ldots,M_r)$) has the following property that the loop channel of the overlap of the crosscap does *not* mix different kinds of characters. This clearly implements the β_r-projection when the k_i are all odd. This will presumably require some modification when even k_i are involved.

5.5.2 The β_0-projection

We still have to implement the projection onto states with integer U(1) charge. This is equivalent to orbifolding by the discrete group $\mathbb{Z}_K$ ($K=\mathrm{lcm}[4,2(k_i+2)]$) generated by $\widehat{g}\equiv[h_0g_1h_1\ldots g_rh_r]$. This symmetry is generated by simple current obtained by tensoring the $(0,2,2)$ fields in the individual minimal models.

The simple way to obtain crosscap states in the orbifold from the unorbifolded theory is to consider the linear combination of all crosscaps that are in a single orbit of the orbifolding group. We naively expect something of the form

$$|C:[\mu],S\rangle \equiv \mathcal{P}_o|C:\mu,S\rangle = \frac{(1+g+\cdots+g^K)}{\sqrt{K}}|C:\mu,S\rangle = \frac{1}{\sqrt{K}}\sum_{\nu_0}^{K-1}|C:\mu{+}\nu_0\mu_0,S{+}2\nu_0\rangle\,. \tag{5.16}$$

where $\mu_0 \equiv (\overbrace{1, \cdots, 1}^{r \text{ times}})$. However, as we have seen, the $\mathbb{Z}_K$ action is projectively realised on crosscap states (see eq. (5.8)). This implies that one needs to introduce suitable phases into the above expression. The orbifolded crosscap state is given by[21]

$$|C:[\mu],S\rangle = \sum_{\nu_0=0}^{K-1} \frac{e^{i\pi\nu_0}}{\sqrt{K}} \left[\frac{\overline{\sigma}_{\mu+2\nu_0\mu_0;S+2\nu_0}}{\overline{\sigma}_{\mu,S}}\right] |C:\mu+2\nu_0\mu_0, S+2\nu_0\rangle \tag{5.17}$$

where $\overline{\sigma}_{\mu,S} \equiv \prod_i \sigma_{0,M_i,S}$. Since all the crosscaps from the individual models have the same S, the spacetime part is uniquely fixed. So the index $(S+2\nu_0)$, indicates that the spacetime part is shifted as well. One can put in the explicit form of the crosscap state in the individual minimal model and see that Ishibashi states whose total U(1) charge is not odd integral are projected out. So the β_0 projection has been carried out.

One can check that the following properties are true (atleast for odd k_i)

$$\begin{aligned} |C:[\mu+2\mu_0],S+2\rangle &= -\left[\frac{\overline{\sigma}_{\mu,S}}{\overline{\sigma}_{\mu+2\mu_0;S+2}}\right]|C:[\mu],S\rangle \\ |C:[\mu],S+4\rangle &= (-)^{Sr}|C:[\mu],S\rangle\,. \end{aligned} \tag{5.18}$$

For even S, the crosscap states consists of NSNS Ishibashi states alone and for odd S, they consist of RR Ishibashi states alone. Clearly, this cannot provide a supersymmetric crosscap. The basic reason is that using either do not produce the produce the full Klein-bottle partition function of a supersymmetric theory — using NSNS part only reproduce the characters without $(-1)^F$-twisting, *viz.* NS and R, while using RR part only reproduce the characters with $(-1)^F$-twisting only, *viz.* $\widetilde{NS}$ and $\widetilde{R}$. Hence, the supersymmetric crosscap is a linear combination of the two possibilities. The spectral flow generator is the product of the spectral flow generators, f_i, in the individual models and let f_0 indicate the same in the spacetime sector. The full crosscap in the Gepner model is given by the orbifolding by the group f. However, since $f^2 = gh$, this has a $\mathbb{Z}_2$ action on the crosscap states already constructed. We obtain

$$|C:[\mu]\rangle_{\text{Gepner}} \equiv \frac{1}{\sqrt{2}}(|C:[\mu],0\rangle \pm |C:[\mu+\mu_0],1\rangle)\,. \tag{5.19}$$

This is our proposal for the crosscap state for Gepner models. The two signs reflect the possibility two having orientifold planes and anti-orientifold planes, since the signs of RR charges are opposite.

6. A toy model: $(k=1)^3$ Gepner model

As a warm-up to orientifold of Calabi-Yau 3-fold we start with the Gepner model realization of the simplest Calabi-Yau, *viz.* CY_1. This is nothing but compactification on T^2 at some special values of its moduli. Let the complex structure and Kähler moduli of T^2 be denoted

[21]The general analysis of crosscap states has been carried in ref. [31] and seems to suggest the addition of additional phases in the crosscap states. However, we have taken the practical view of looking to directly implement in the β_0 projection, i.e., keep only Ishibashi states that have odd integral U(1) charge.

by τ and ρ. Since compactification on T^2 corresponds to $c = 3$, it can be realized as 3 types of Gepner models [63], *viz.* $(k = 1)^3$, $(k = 2)^2$ and $(k = 1) \cdot (k = 4)$. All these Gepner models correspond to T^2 compactification at the enhanced symmetry points, e.g., the former corresponds to T^2 compactification at SU(3) point with $(\tau,\rho) = (e^{2\pi i/3}, e^{2\pi i/3})$, the second on T^2 at the SU(2)^2 point with $(\tau, \rho) = (i, i)$ and the latter on T^2 at SU(3) point with $(\tau, \rho) = (e^{2\pi i/3}, e^{2\pi i/3})$. For simplicity we consider the $(k = 1)^3$ Gepner model which is equivalent to compactification on an SU(3) torus with $(\tau, \rho) = (e^{2\pi i/3}, e^{2\pi i/3})$; the SU(3) torus is generated by quotienting $\mathbf{R}^2$ by SU(3) root lattice. We now discuss the Gepner model and construction of crosscap states therein.

6.1 A-type crosscap states of $(k = 1)^3$ model

In this section we discuss the orientifold of $(k = 1)^3$ Gepner model. We consider type-IIA theory compactified on this Gepner model. We discuss the symmetry group of this model and construct the orientifold model from RCFT approach. Our main goal is to write all possible crosscap states in this orientifold model and compute their overlaps and hence the KB amplitudes. Using the abstract formalism of section 2, we have already written down the formula for crosscaps in Gepner model in section 5. At the final step, we need to know the precise orientifold group for the model. This allows us to consider crosscaps arising from the simple currents. These simple currents are due to the presence of discrete orbifold group present in the full orientifold group. We discuss this issue in the next section.

6.1.1 The crosscaps

Once we know the orientifold group of the theory, it is quite straightforward to write down the crosscap states. Since for $(k = 1)^3$ model we know the orientifold group is as given by eq. (5.11), we immediately infer that this model has 9 crosscap states. In the large volume limit, when we have the geometric phase of Calabi-Yau, these crosscaps correspond to $\mathbb{RP}^1$ cycles.

We first write down the NSNS part of the crosscaps in the theory. As prescribed in eq. (4.16), we set $L_1 = L_2 = L_3 = 0$. To determine $[\mu] = (M_1, M_2, M_3)$ in eq. (5.19), we apply the machinery of section 4.2. The simple currents which generate these 9 crosscaps are obtained by tensoring the currents Φ_{000}, $(\Phi_{011})^2 = \Phi_{022}$ and $(\Phi_{011})^4 = \Phi_{044}$ of the $(k = 1)^3$ theory.[22] Thus NSNS part of the crosscap state of this model can be denoted as

$$|C : M_r = 2a_r\rangle_{NSNS} \equiv |C : [\mu] = [2a]\rangle_{NSNS}\,, \qquad a_r = 0, 1, 2\,. \tag{6.1}$$

After obtaining the NSNS part, it is now easy to get the RR part. The general prescription is to spectral flow the NSNS boundary state by f, as we discussed in section 5.2. So following eq. (5.19), the total crosscap state in the Gepner model $(k = 1)^3$ is

$$|C\,[\mu]\,;\,(k = 1)^3\rangle_{\text{Gepner}} = \frac{1}{\sqrt{2}}\Big[|C\,[\mu] = [2a]\rangle_{\text{NSNS}} + |C\,[\mu + \mu_0], 1\rangle_{\text{RR}}\Big]\,. \tag{6.2}$$

[22] It can be obtained by another way. As we see from eq. (5.19) and section 4.2, we set $L_i = S_i = 0$ for these states. The only relevant labels for these states are M_i which take values $0, 2, 4 \mod 6$ i.e. 27 values. It becomes 9 after imposing the equivalence $\mu \leftrightarrow \mu + 2 + \mu_0$.

6.2 Spectra, character formulae and spectral flow invariant orbits of $k = 1$ minimal model

Before we go and discuss the KB amplitude in this model, we need to know the spectra of the $k = 1$ minimal model and how to express it neatly in terms of spectral flow invariant orbits of supersymmetric characters — a technique introduced in ref. [44]. This is a very powerful technique for discussing Gepner model and in fact, we shall express the KB amplitude in this model in terms of these orbits.

	l	m	h	q
A_+	0	0	0	0
B_+	1	1	$\frac{1}{6}$	$\frac{1}{3}$
C_+	1	-1	$\frac{1}{6}$	$-\frac{1}{3}$

Table 3: NS representations of the $k = 1$ minimal model and their characters.

The NS-sector of $k = 1$ minimal model contains three representations as given in table 3 along with their conformal weights and U(1) charge.

Under $\eta \to \eta + 1$, the spectral flow amongst the three types of characters are given by

$$A_\pm \to B_\pm \to C_\pm \to A_\pm\,. \tag{6.3}$$

The formulae of the NS- and R-sector characters in terms of the string functions and theta functions can be found in eqs. (E.3), (E.7) and (E.8) in the appendix E.

The spectral flow invariant orbits for $(k = 1)^3$ are really very trivial [44, 38]. This model has the following two spectral flow invariant orbits in NS sectors:

1. "Massless Orbit":
$$\mathcal{NS}_0^+ = A_+^3 + B_+^3 + C_+^3\,. \tag{6.4}$$
We call this orbit the "massless" orbit, since it gives rise to graviton($h = q = 0$) and massless matters($h = 1/2$, $q = \pm 1$) in spacetime.

2. "Massive Orbit":
$$\mathcal{NS}_1^+ = 3\,A_+\,B_+\,C_+\,. \tag{6.5}$$
This orbit is called the "massive" orbit, since it gives rise to massive matter in spacetime.

3. There are two other orbits corresponding to $\mathcal{NS}_0^+$ and $\mathcal{NS}_1^+$ which appear in the $(-1)^F$ twisted sectors:
$$\mathcal{NS}_0^- = A_-^3 + B_-^3 + C_-^3\,, \qquad \mathcal{NS}_1^- = 3\,A_-B_-C_-\,. \tag{6.6}$$

6.3 KB amplitude

We now compute the KB amplitude in our model using the ansatz given in eqs. (5.17) and (5.19). It is given by sum of two pieces: an NSNS amplitude and a RR amplitude;

$$\begin{aligned}\langle C\,:\,[\widetilde{\mu}]|e^{\frac{-\pi i}{2\tau}H_{cl}}\,|C\,:\,[\mu]\rangle = {}_{\mathrm{NSNS}}\langle C\,:\,[\widetilde{\mu}] = [2\widetilde{a}], 0|e^{\frac{-\pi i}{2\tau}H_{cl}}\,|C\,:\,[\mu] = [2a], 0\rangle_{\mathrm{NSNS}} + \\ + {}_{\mathrm{RR}}\langle C\,;\,[2\widetilde{a} + \mu_0], 1|e^{\frac{-\pi i}{2\tau}H_{cl}}\,|C\,:\,[2a + \mu_0]\rangle_{\mathrm{RR}}\,.\end{aligned} \tag{6.7}$$

NSNS part of the KB amplitude: Using the identity $Y^0_{l_i\ 0} = (-1)^{l_i}$, the NSNS amplitude is given by

$$\begin{aligned}\mathcal{K}_{\rm NSNS} &= {}_{\rm NSNS}\langle C\,:\,[2\widetilde{a}],0|e^{-\frac{\pi i H}{2\tau}}|C\,:\,[2a],0\rangle_A \\ &= \frac{1}{K}\sum_{\tilde\nu_0=0}^{K-1}\sum_{\nu_0=0}^{K-1}\delta^2_{[\mu]+[\tilde\mu]}e^{-i\pi\nu_0}e^{+i\pi\tilde\nu_0}\overline{\sigma}_{0,2\tilde a,0}\overline{\sigma}_{0,2a,0}\frac{1}{2}(\chi_{\rm st}^{-\nu_0}+\chi_{\rm st}^{-\nu_0+2})\times \\ &\quad\times\prod_i\sum_{l_i}^{\rm ev}(-)^{l_i}\left[\chi^{(-(\nu_0-\tilde\nu_0))}_{l_i,\frac{\tilde M_i-M_i}{2}-(\nu_0-\tilde\nu_0)}(2\tau)+\chi^{(-(\nu_0-\tilde\nu_0)+2)}_{l_i,\frac{\tilde M_i-M_i}{2}-(\nu_0-\tilde\nu_0)}(2\tau)\right], \qquad (6.8)\end{aligned}$$

where $\chi_{\rm st}$ is the appropriate spacetime character. In evaluating the sum over ν_0 and $\tilde\nu_0$, it is better to to shift ν_0 by $\tilde\nu_0$, this shift does not make any difference since the arguments are periodic.

The final form of NSNS part of the KB amplitude is

$$\begin{aligned}\mathcal{K}_{\rm NSNS} &= {}_{\rm NSNS}\langle C\,:\,[2\widetilde{a}],0|e^{-\frac{\pi i H}{2\tau}}|C\,:\,[2a],0\rangle_{\rm NSNS} \\ &= \sum_{\nu_0=0}^{K-1} e^{-i\pi\nu_0}\delta^2_{[\mu]+[\tilde\mu]}\overline{\sigma}_{0,2\tilde a,0}\overline{\sigma}_{0,2a,0}\frac{1}{2}(\chi_{\rm st}^{-\nu_0}+\chi_{\rm st}^{-\nu_0+2})\times \\ &\quad\times\prod_i\sum_{l_i}^{\rm ev}(-)^{l_i}\left[\chi^{(-\nu_0)}_{l_i,\frac{\tilde M_i-M_i}{2}-\nu_0}+\chi^{(-\nu_0+2)}_{l_i,\frac{\tilde M_i-M_i}{2}-\nu_0}\right](2\tau)\,. \qquad (6.9)\end{aligned}$$

Eq. (6.9) is the *master formula* for KB amplitude in NSNS sector. The above formula holds in general (as long as all the k_i are odd) and thus holds for the $(k=1)^3$ and $(k=3)^5$ Gepner models. Thus, we have kept eq. (6.9) completely generic. Here $K = \mathrm{lcm}\,(4\,,\,2\,k{+}4)$ and p is the number of minimal models being tensored. For $k=1$, $K=12$ and $p=3$.

Using the formulae (6.4), (6.5) for the flow invariant orbits, we can finally express the KB amplitudes for crosscaps for $(k=1)^3$ Gepner model in terms these orbits. The answer for the case of amplitudes involving the same crosscap state is

$$\begin{aligned}&{}_{\rm NSNS}\langle C\,:\,[\mu]=[2a]|e^{\frac{-\pi i}{2\tau}H_{cl}}|C\,:\,[\mu]=[2a]\rangle_{\rm NSNS} = \\ &= \Bigg[\chi_{\rm st}^{+,({\rm NS})}\Bigg(\prod_{r=1}^3(\chi^0_{00}+\chi^2_{00})(2\tau)+\prod_{r=1}^3(\chi^0_{11}+\chi^2_{11})(2\tau)+\prod_{r=1}^3(\chi^0_{1-1}+\chi^2_{1-1})(2\tau)\Bigg)- \\ &\quad -\chi_{\rm st}^{+,({\rm R})}\Bigg(\prod_{r=1}^3(\chi^1_{01}+\chi^{-1}_{01})(2\tau)+\prod_{r=1}^3(\chi^1_{10}+\chi^{-1}_{10})(2\tau)+\prod_{r=1}^3(\chi^1_{12}+\chi^{-1}_{12})(2\tau)\Bigg)\Bigg] \\ &= \Big[\chi_{\rm st}^{+,({\rm NS})}\Big(A^3_++B^3_++C^3_+\Big)-\chi_{\rm st}^{+,({\rm R})}\Big(\widehat{A}^3_++\widehat{B}^3_++\widehat{C}^3_+\Big)\Big](2\tau)\,, \qquad (6.10)\end{aligned}$$

Thus the NS-part of the KB amplitudes in $(k=1)^3$ Gepner model is schematically(i.e. suppressing the Minimal model labels)

$$\begin{aligned}\mathrm{KB}^{(k=1)^3}_{\rm NS} &= \chi_{\rm st}^{+,({\rm NS})}(A^3_++B^3_++C^3_+)(2\tau)-\chi_{\rm st}^{+,({\rm R})}(\widehat{A}^3_++\widehat{B}^3_++\widehat{C}^3_+)(2\tau) \\ &= \chi_{\rm st}^{+,({\rm NS})}\,\mathcal{NS}_0^+(2\tau)-\chi_{\rm st}^{+,({\rm R})}\,\mathcal{R}_0^+(2\tau)\,. \qquad (6.11)\end{aligned}$$

This eq. (6.11) is the desirable result, as it agrees with the general structure of the NSNS part of the KB amplitude. As discussed after eq. (5.5), the NSNS part of crosscap state reproduces the characters NS and R in the loop channel, i.e.

$$\begin{aligned} {}_{\rm NSNS}\langle C\,;[\mu]|e^{\frac{-\pi i}{2\tau}H_{cl}}|C\,[\mu]\rangle_{\rm NSNS} &= \frac{1}{2}\Big[\mathrm{Tr}_{\rm NSNS}\Big[\Omega\cdot\sigma\cdot\mathsf{P}e^{2\pi i\tau\,H_{cl}}\Big]-\mathrm{Tr}_{\rm RR}\Big[\Omega\cdot\sigma\cdot\mathsf{P}\cdot e^{2\pi i\tau\,H_{cl}}\Big]\Big] \\ &\equiv NS-R\,. \end{aligned} \tag{6.12}$$

Thus from eqs. (6.11) and (6.12), we find that for overlaps of crosscaps,

$$NS=\chi_{\rm st}^{+,({\rm NS})}\,\mathcal{NS}_0^+(2\tau)\,,\qquad R=\chi_{\rm st}^{+,({\rm R})}\,\mathcal{R}_0^+(2\tau) \tag{6.13}$$

Here P denote the projector for the subgroup $\mathcal{H}=\mathbb{Z}_3^3/\mathbb{Z}_3$. The sum $(-1)^F$ twisted characters, actually comes from the overlap of the RR-part of the crosscap state [19].

RR part of the KB amplitude: we can repeat the same analysis for the RR sector amplitude. We find

$$\begin{aligned} \mathrm{KB}_{\rm RR} &= {}_{\rm RR}\langle C\ :[2\widetilde{a}+\mu_0],1|e^{\frac{-\pi i}{2\tau}H_{cl}}|C\,[2a+\mu_0],1\rangle_{\rm RR} \\ &= \Big[\chi_{\rm st}^{-,({\rm NS})}\Big(A_-^3+B_-^3+C_-^3\Big)-\chi_{\rm st}^{-,({\rm R})}\Big(\widehat{A}_-^3+\widehat{B}_-^3+\widehat{C}_-^3\Big)\Big](2\tau) \\ &= \chi_{\rm st}^{-,({\rm NS})}\,\mathcal{NS}_0^-(2\tau)-\chi_{\rm st}^{-,({\rm R})}\,\mathcal{R}_0^-(2\tau)\,. \end{aligned} \tag{6.14}$$

Since we expect

$$\begin{aligned} {}_{\rm RR}\langle C\ :[2\widetilde{a}+\mu_0],1|e^{\frac{-\pi i}{2\tau}H_{cl}}|C\,[2a+\mu_0],1\rangle_{\rm RR} &= \frac{1}{2}\Big[\mathrm{Tr}_{\rm NSNS}[\Omega\cdot\sigma\cdot\mathsf{P}\,(-1)^F\,e^{2\pi i\tau\,H_{cl}}]- \\ &\quad -\mathrm{Tr}_{\rm RR}[\,\Omega\cdot\sigma\cdot\mathsf{P}\cdot(-1)^F\cdot e^{2\pi i\tau\,H_{cl}}]\Big] \\ &\equiv \widetilde{NS}-\widetilde{R}\,, \end{aligned} \tag{6.15}$$

in the loop channel, we find from eqs. (6.13) and (6.15) that for RR overlaps between crosscaps of this Gepner model

$$\widetilde{NS}=\chi_{\rm st}^{-,({\rm NS})}\,\mathcal{NS}_0^-(2\tau)\,,\qquad \widetilde{R}=\chi_{\rm st}^{-,({\rm R})}\mathcal{R}_0^-(2\tau)\,. \tag{6.16}$$

Summing eqs. (6.11) and (6.14) we get the total KB amplitude in the orientifold of $(k=1)^3$ Gepner model:

$$\begin{aligned} \mathrm{KB}_{\rm total}^{(k=1)^3} &= (NS+\widetilde{NS})-(R+\widetilde{R}) \qquad (6.17)\\ &= \Big(\chi_{\rm st}^{+,({\rm NS})}\mathcal{NS}_0^+ +\chi_{\rm st}^{-,({\rm NS})}\mathcal{NS}_0^-\Big)(2\tau)-\Big(\chi_{\rm st}^{+,({\rm R})}\mathcal{R}_0^+ +\chi_{\rm st}^{-,({\rm R})}\mathcal{R}_0^-\Big)(2\tau)\,. \end{aligned}$$

The result involving different crosscaps which however preserve the same supersymmetry is non-zero and involves the *massive* orbits. In this example, it is $3ABC$ and characters obtained from this by means of spectral flow.

JHEP02(2004)026

7. KB amplitudes in $(k=3)^5$ Gepner model

Finally we come to the point of computing the KB amplitudes in the orientifolds of the quintic. As discussed in section 5, compactification on the quintic hypersurface can be represented by tensoring five copies of $k=3$ minimal model. This is Gepner's famous $(k=3)^5$ model [47]. The formula for the KB amplitudes can be written straight away using the master formula eq. (6.11) derived in section 6.3 by substituting the relevant values for p and K. Next problem is to manipulate this expression so as to express it in terms of orbits of $k=3$ minimal model and we can derive something meaningful from it. Our previous experience with $(k=1)^3$ model shows that the answer should be proportional to the massless orbits.This is also expected from hindsight, since only the massless tadpoles flow in the loop channel.

So we need to know the representations of $k=3$ minimal model. This can be worked out along the lines of $k=1$ model. The only non-trivial part is to work out the spectral flow invariant orbits which we will discuss here. To work out these orbits, it is quite useful to see what $(2,2)$ SCFT teaches us about this spectra and the massless and massive representation. Note that before taking the orientifold projection the worldsheet algebra was given by $(2,2)$ SCFT.

7.1 (2,2) SCFT spectrum

To work out the spectrum for compactification with (2,2) SCFT, the knowledge of $\mathcal{N}=2$ SCA is not enough. In fact, we already noticed that this SCA has an automorphism generated by the spectral flow operator U_{η_L,η_R}, eq. (3.3). It is well-known that under the spectral flow, NS- and R-sectors are transformed into each other, so that the spacetime bosons and fermions are paired. To work out the massless and massive spectrum of (2,2) SCFT with $c=9$, one has to incorporate the spectral flow operator into the $\mathcal{N}=2$ SCA [44]. This makes the latter into a bigger algebra; it is not of the kind of Lie algebra but a kind of W-algebra. This algebra can be worked out as follows. Representations of $\mathcal{N}=2$ SCA are labeled by the conformal weight h and the U(1) charge q. In the case of $c=3n$, where c is the central charge of the Virasoro algebra, the vacuum state($h=0$, $q=0$) in the NS sector is mapped onto the states ($h=n/8$, $q=\pm n/2$) in the R-sector and ($h=n/2$, $q=\pm n$) in the NS-sector under the spectral flow $\eta=\pm 1/2\,,\pm 1$ respectively. The states with ($h=n/8$, $q=\pm n/2$) and ($h=n/2$, $q=\pm n$) are respectively the covariantly constant spinors and (anti-)holomorphic n-forms of CY_n with $\mathrm{SU}(n)$ holonomy. The former corresponds to the spacetime supersymmetry and the latter to the flow generators [44]. We denote the operators corresponding to flow generators by $\mathrm{U}(z)$ and $\overline{U}(\bar{z})$ and its superpartners with $(h,\, q)$ $= (\frac{n+1}{2},\, \pm(n-1))$ by $V(z)$ and $\overline{V}(\bar{z})$. For compactification on CY_3, we have $c=9$ hence $n=3$ and the flow generators are states labeled by ($h=3/2$, $q=\pm 3$).

This bigger algebra had been worked out for $c=9$ by Odake [45]. We call it *Odake Algebra* and list the values of h and q_L for left-moving massless and massive spectra in table 4. Similarly for right-moving sectors.

It shows that there are mainly 3 massless and 2 massive orbits in $c=9$ SCFT. It helps us immensely in constructing the orbits for $(k=3)^5$ Gepner model. We must make a note

	NS	R
	$\widehat{\mathrm{NS}}_1$: $h = 0$, $q_L = 0$	$\widehat{\mathrm{R}}_1$: $h = \frac{3}{8}$, $q_L = \pm\frac{3}{2}$ (Identity operator)
Massless	$\widehat{\mathrm{NS}}_2$: $h = \frac{1}{2}$, $q_L = 1$	$\widehat{\mathrm{R}}_2$: $h = \frac{3}{8}$, $q_L = -\frac{1}{2}$ $(\eta_L = -\frac{1}{2})$
	$\widehat{\mathrm{NS}}_3$: $h = \frac{1}{2}$, $q_L = -1$	$\widehat{\mathrm{R}}_3$:$h = \frac{3}{8}$, $q_L = \frac{1}{2}$ $(\eta_L = \frac{1}{2})$
Massive	$\widehat{\mathrm{NS}}_4$: $h > 0$, $q_L = 0$	$\widehat{\mathrm{R}}_4$: $h > \frac{3}{8}$, $q_L = \pm\frac{3}{2}, \pm\frac{1}{2}$
	$\widehat{\mathrm{NS}}_5$: $h > \frac{1}{2}$, $q_L = \pm 1$	$\widehat{\mathrm{R}}_5$: $h > \frac{3}{8}$,$q_L = \pm\frac{1}{2}$

Table 4: Dimensions and U(1) charges of the massless and massive spectra of the holomorphic part of the (2,2) internal SCFT. Similar results holds for the antiholomorphic part.

of one point. The Odake algebra gives rise to an *irrational* CFT, since it contains *infinite* number of primaries. But the same $c = 9$ SCFT can be constructed as an RCFT, as e.g., in terms of $(k = 3)^5$ Gepner model. So one might wonder about the connection between the Gepner model and the abstract $c = 9$ Odake algbera and its representation. In fact, if we perform an infinite sum over the characters of the latter, that organise itself nicely into the orbits of Gepner model. This point was emphasized in ref. [44, 45] and they explicitly checked it for a few other Gepner models.

7.2 Spectra and spectral flow invariant orbits of $k = 3$ minimal model

The NS-sector of $k = 3$ minimal model consists of ten representations as given in table 5 of appendix F. We also provide their conformal weights and U(1) charges. From the table 5, we find that ten representations of $k = 3$ minimal model forms two non-overlapping groups of five each under the spectral flow $\eta \to \eta + 1$. They are

$$A_+ \to J_+ \to I_+ \to H_+ \to G_+ \to A_+\,, \tag{7.1}$$

$$B_+ \to F_+ \to E_+ \to D_+ \to C_+ \to B_+\,. \tag{7.2}$$

Labels for the characters	l	m	h	q	Spectral flow $(\eta \to \eta + 1)$
$A_\pm$	0	0	0	0	$A_\pm \to J_\pm$
$B_\pm$	1	-1	1/10	$-1/5$	$B_\pm \to F_\pm$
$C_\pm$		1	1/10	1/5	$C_\pm \to B_\pm$
$D_\pm$		-2	1/5	$-2/5$	$D_\pm \to C_\pm$
$E_\pm$	2	0	2/5	0	$E_\pm \to D_\pm$
$F_\pm$		2	1/5	2/5	$F_\pm \to E_\pm$
$G_\pm$		-3	3/10	$-3/5$	$G_\pm \to A_\pm$
$H_\pm$	3	-1	7/10	$-1/5$	$H_\pm \to G_\pm$
$I_\pm$		1	7/10	1/5	$I_\pm \to H_\pm$
$J_\pm$		3	3/10	3/5	$J_{pm} \to I_\pm$

Table 5: The representations and their spectral flow for $k = 3$ minimal model.

JHEP02(2004)026

The expressions of $k = 3$ NS-sector characters can be found in eqs. (F.5) in appendix F.

We now write down the most obvious four spectral flow invariant orbits in the NS sector of the model.

1. "Graviton Orbit"

$$\mathcal{NS}_0^+ = A_+^5 + G_+^5 + H_+^5 + I_+^5 + J_+^5 \,. \tag{7.3}$$

   It contains the operator for the spacetime graviton($h = q = 0$).

2. "Self-conjugate Massless Matter Orbit"

$$\mathcal{NS}_1^+ = B_+^5 + C_+^5 + D_+^5 + E_+^5 + F_+^5 \,. \tag{7.4}$$

   It contains the operators for both hyper and vector multiplets($h = 0$, $q = \pm 1$) and hence the name.

3. "Massive Orbits"

$$\mathcal{NS}_2^+ = 5\, A_+ \, G_+ \, H_+ \, I_+ \, J_+ \,, \qquad \mathcal{NS}_3^+ = 5\, B_+ \, C_+ \, D_+ \, E_+ \, F_+ \,. \tag{7.5}$$

   Both NS_2 and NS_3 are "massive" orbits.

4. There are $(-1)^F$ twisted orbits corresponding to $\mathcal{NS}_0^+$, $\mathcal{NS}_1^+$, $\mathcal{NS}_2^+$ and $\mathcal{NS}_3^+$. They are

$$\begin{aligned} \mathcal{NS}_0^- &= A_-^5 + G_-^5 + H_-^5 + I_-^5 + J_-^5 \\ \mathcal{NS}_1^- &= B_-^5 + C_-^5 + D_-^5 + E_-^5 + F_-^5 \\ \mathcal{NS}_2^- &= 5\, A_- G_- H_- I_- J_- \,, \qquad \mathcal{NS}_3^- = 5\, B_- C_- D_- E_- F_- \,. \end{aligned} \tag{7.6}$$

These are not the set of all orbits for this model. There are other massive orbits. Since we do not require them, we are not listing them here.

The orbits in R-sector of this model, in particular, $\mathcal{R}_0^\pm$, $\mathcal{R}_1^\pm$, $\mathcal{R}_2^\pm$, $\mathcal{R}_3^\pm$ and $\mathcal{R}_4^\pm$ are obtained from eqs. (7.3), (7.4), (7.5) and (7.6) by replacing the supersymmetric characters by their hatted counterpart as defined in eq. (F.7) in appendix F.

7.3 KB amplitudes

In this section we are going to compute the KB amplitudes for $(k = 3)^5$ Gepner model corresponding to quintic. Since the orientifold group for this model is as given in eq. (5.11),[23] clearly it gives rise to $5^4 = 625$ crosscaps states which in the large volume limit corresponds to 625 $\mathbb{RP}^3$'s. In the Gepner model such crosscaps are constructed as follows. All such 625 crosscaps are actually generated by the symmetry group elements of $\mathcal{H}$ from the basic one i.e. the crosscap with $L_i = M_i = S_i = 0$. So we can apply the techniques of section 4.2, in particular, we apply the formulae (4.19) and (4.20). It is easy to figure out that these 625 crosscaps are generated by tensoring the simple currents Φ_{000},

[23] Compared to $(k = 1)^3$ model, for this model we need the the factor of $(-1)^{F_L}$ in the orientifold group.

$(\Phi_{011})^2 = \Phi_{022}$, $(\Phi_{011})^4 = \Phi_{044}$, $(\Phi_{011})^6 = \Phi_{066}$ and $(\Phi_{011})^8 = \Phi_{088}$. It implies that we should put $M_i = 0, 2, 4, 6, 8 \mod 10$ in eq. (5.1) or eq. (5.3) in each crosscap state of five copies of minimal model. Thus the NSNS part of a generic crosscap state(A-type) in this model would be

$$\Big|C\,;\,[\mu] = [2a]\Big\rangle_{\rm NSNS}, \quad a = 0, 2, 4, 6, 8 \mod 10 \tag{7.7}$$

where $a_r = 0, 2, 4, 6, 8$. The RR part of the crosscap state is

$$\Big|C\,;\,[\mu] = [2a], 1\Big\rangle_{\rm RR}, \quad a = 0, 2, 4, 6, 8 \mod 10\,. \tag{7.8}$$

The full crosscap state of the Gepner model is the sum of the NSNS part and the RR part with Gepner projection applied as in eqs. (5.17) and (5.19).

All remains is to compute the KB amplitude using the formulae for NSNS and RR part of the crosscaps. The master formula had already been worked out in eq. (6.9). Since we have found the flow invariant orbits of the model, it is easy to write down the formula for the KB amplitude in the orientifold of $(k = 3)^5$ Gepner model. Putting $p = 5$ and $K = 20$ in eq. (6.9) and explicitly evaluating it, we find that for $\mu = \tilde{\mu}$ we get[24]

$$\begin{aligned}
\mathrm{KB}_{\rm NS}^{(k=3)^5} &= NS - R \\
&= \Big[\frac{1}{2}\chi_{\rm st}^{+,({\rm NS})}\Big\{(A_+^5 + G_+^5 + H_+^5 + I_+^5 + J_+^5) + (B_+^5 + C_+^5 + D_+^5 + E_+^5 + F_+^5)\Big\}(2\tau) - \\
&\quad - \frac{1}{2}\chi_{\rm st}^{+,({\rm R})}\Big\{(\widehat{A}_+^5 + \widehat{G}_+^5 + \widehat{H}_+^5 + \widehat{I}_+^5 + \widehat{J}_+^5) + (\widehat{B}_+^5 + \widehat{C}_+^5 + \widehat{D}_+^5 + \widehat{E}_+^5 + \widehat{F}_+^5)\Big\}(2\tau)\Big] \\
&= \chi_{\rm st}^{+,({\rm NS})}(\mathcal{NS}_0^+ + \mathcal{NS}_1^+) - \chi_{\rm st}^{+,({\rm R})}(\mathcal{R}_0^+ + \mathcal{R}_1^+)\,.
\end{aligned} \tag{7.9}$$

This is the desired result, since it shows that only the orbits of massless tadpoles circulate in the loop channel. Similarly the RR amplitudes for crosscaps satisfying $[\mu] = [\widetilde{\mu}]$ is

$$\begin{aligned}
\mathrm{KB}_{\rm RR}^{(k=3)^5} &= \widetilde{NS} - \widetilde{R} \\
&= \Big[\frac{1}{2}\chi_{\rm st}^{-,({\rm NS})}\Big\{(A_-^5 + G_-^5 + H_-^5 + I_-^5 + J_-^5) + (B_-^5 + C_-^5 + D_-^5 + E_-^5 + F_-^5)\Big\}(2\tau) \\
&\quad - \frac{1}{2}\chi_{\rm st}^{-,({\rm R})}\Big\{(\widehat{A}_-^5 + \widehat{G}_-^5 + \widehat{H}_-^5 + \widehat{I}_-^5 + \widehat{J}_-^5) + (\widehat{B}_-^5 + \widehat{C}_-^5 + \widehat{D}_-^5 + \widehat{E}_-^5 + \widehat{F}_-^5)\Big\}(2\tau)\Big] \\
&= \chi_{\rm st}^{-,({\rm NS})}\,(\mathcal{NS}_0^- + \mathcal{NS}_1^-)\ -\ \chi_{\rm st}^{-,({\rm R})}\,(\mathcal{R}_0^- + \mathcal{R}_1^-)
\end{aligned} \tag{7.10}$$

So adding eqs. (7.9) and (7.10), we find the total KB amplitude in the orientifold of quintic[25]

$$\mathrm{KB}_{\rm total}^{(k=3)^5} = (NS + \widetilde{NS}) - (R + \widetilde{R})$$

[24]Here we define $NS = \frac{1}{2}\,\mathrm{Tr}_{\rm NSNS}[\Omega\cdot(-1)^{F_L}\cdot\sigma\cdot\mathsf{P}\cdot e^{2\pi i\tau H}]$ and $\widetilde{NS} = \frac{1}{2}\,\mathrm{Tr}_{\rm NSNS}[\Omega\cdot(-1)^{F_L}\cdot\sigma\cdot\mathsf{P}\cdot(-1)^F\cdot e^{2\pi i\tau H}]$; similarly the R-sector characters.

[25]In a supersymmetric theory the NSNS and RR part of a KB amplitude cancel each other and hence the total KB amplitude vanishes identically. We expect that both eqs. (6.17) in section 6.3 and (7.11) above vanish, though proving it requires certain non-trivial identities involving Jacobi's ϑ-functions.

$$= \Big(\chi_{\rm st}^{+,({\rm NS})}(\mathcal{NS}_0^+ + \mathcal{NS}_1^+) + \chi_{\rm st}^{-,({\rm NS})}(\mathcal{NS}_0^- + \mathcal{NS}_1^-)\Big)(2\tau)$$
$$-\Big(\chi_{\rm st}^{+,({\rm R})}\ (\mathcal{R}_0^+ + \mathcal{R}_1^+)\ +\ \chi_{\rm st}^{-,({\rm R})}\ (\mathcal{R}_0^- + \mathcal{R}_1^-)\Big)(2\tau)\,. \qquad (7.11)$$

As in case of $(k=1)^3$ model, the *different* crosscaps preserving *same* supersymmetry gives rise to *massive* characters.

From eq. (7.11), we find that the characters $(\mathcal{NS}_0^+ \ + \ \mathcal{NS}_1^+)$ appear with a plus sign, which implies implies that the symmetric parts of the NSNS sectors are projected in. Note that $\mathcal{NS}_0^+$ is the graviton sector and this is correct. $\mathcal{NS}_1^+$ is the vector multiplet and this implies that the scalars that appear from the NSNS sector are projected in. The characters $(\mathcal{R}_0^+ \ + \ \mathcal{R}_1^+)$ in eq. (7.11) appear with a minus sign. This implies that the graviphoton and the vector in vector multiplet are projected out. This is identical to what we found in section 3.

The next step would be check the conditions that are obtained from tadpole cancellation. In this regard, we observe that the $(1,1,1,1,1)$ Recknagel-Schomerus states are the D-brane boundary states that should be added to obtain vanishing of tadpoles. One can see that this is indeed possible by studying the corresponding annulus amplitudes.

8. Conclusion

In this paper we have studied the orientifolds of Calabi-Yau manifolds at the Gepner point using techniques of rational conformal field theory. In particular, we have constructed the crosscaps for quintic. We computed the Klein bottle amplitudes using these crosscaps for quintic. The result we obtained from this abstract RCFT techniques has been verified against the geometric as well as SCFT results. To our satisfaction, they agree.

Obviously, this is only a first step. We did not discuss the important unoriented open string sector which has to be included to cancel RR tadpoles and extraction of the spectra. This can be extracted from the Möbius strip amplitudes. This will be discussed in our next paper [74]. Other issues that we hope to address include the case of even k (not considered here for simplifying the analysis), computation of intersection matrices via a Witten index computation, extraction of RR charges. The case of even k is interesting in that it has a richer structure of anti-holomorphic involutions. This might lead to the appearance of chiral fermions in the spectra in addition to non-abelian gauge symmetry.

The case of type-IIB orientifolds is interesting as well. In this case, one deals with holomorphic involutions whose fixed points are holomorphic submanifolds. The Kähler moduli which are complex become real after orientifold projection. This implies that the large-volume CY_3, say the quintic, may be separated from the Gepner model by the conifold singularity (there will be no way to "go around" the singularity. At large volume, D-branes are related to coherent sheaves, in general. Orientifolding at large volume, involves extending the notion of duals of vector bundles to sheaves as well. The use of π-stabilty to discuss the stability of D-branes [75] in the orientifold will require some modification. This has to take into account that an unstable brane in the CY_3 case may become stable because its decay product is projected out in the orientifold. This issues will be discussed

JHEP02(2004)026

in a forthcoming paper [80]. The obvious extension of our construction to the case of B-type crosscaps will also be discussed in the future. In fact, one can use the Greene-Plesser construction [76], where the mirror CY_3 is obtained by further orbifolding of the Gepner model, to easily write the B-type crosscap state by a trivial use of the orbifolding procedure of [31] on the crosscaps that we have constructed.

Acknowledgments

We acknowledge discussions with B. Acharya, I. Brunner, M. R. Douglas, B. Florea, C. Hofman, T. Jayaraman and H. Liu. S.G. would like to thank the New High Energy Theory Center of Rutgers University for the invitation as well as hospitality during Summer 2002 during which this work was started and the Theory Group at CERN where the manuscript was completed. We also would like to thank the organisers of PASCOS '2003 where these results were first reported [79].

A. Notation for boundary and crosscap states in RCFT

Let J_n, $\overline{J}_n$ denote the modes of chiral and anti-chiral currents. then the definitions of boundary and crosscap states are:

$$\begin{aligned} \big(J_n + (-1)^{h_J}\, \overline{J}_n \big)|B\rangle &= 0\,, \qquad \text{Boundary} \\ \big(J_n + (-1)^{h_J+n}\, \overline{J}_n \big)|C\rangle &= 0\,, \qquad \text{Crosscap}\,. \end{aligned} \tag{A.1}$$

A basis for the solutions to eq. (A.1) are formed by *Ishibashi states*. They are given by

$$\begin{aligned} \text{Boundary} : |B_i\rangle\rangle &= \sum_I |I\rangle_i \otimes U_B\, |I\rangle_{i^c} \\ \text{Crosscap} : |C_i\rangle\rangle &= \sum_I |I\rangle_i \otimes U_C\, |I\rangle_{i^c}\,, \end{aligned} \tag{A.2}$$

where i labels a representation of the chiral algebra and i^c its charge conjugate. The sum is over all states in a given representation. U_B and U_C are operators which satisfy:

$$\overline{J}_n\, U_B = (-1)^{h_J}\, U_B\, \overline{J}_n\,; \qquad \overline{J}_n\, U_C = (-1)^{h_J+n}\, U_C\, \overline{J}_n \tag{A.3}$$

Any boundary or crosscap state must be a linear combination of these Ishibashi states:

$$|B_a\rangle = \sum_i B_{a\,i}\, |B_i\rangle\rangle\,; \qquad |C\rangle = \sum_i \Gamma_i\, |C_i\rangle\rangle\,. \tag{A.4}$$

B. Constraints on open string amplitudes

- The coefficients $\mathcal{M}^i_{\ a}$ satisfy extra condition [43], *viz.*

$$\left| \mathcal{M}^i_{\ a} \right| \leq \mathcal{A}^i_{\ a\,a}\,, \quad \mathcal{M}^i_{\ a} = \mathcal{A}^i_{\ a\,a} \mod 2\,. \tag{B.1}$$

  This ensures that the open string sector has non-negative, integer state degeneracies in the sum $\frac{(\mathcal{A}+\mathcal{M})}{2}$.

- *Completeness condition for the coefficients of Annulus amplitudes.*

  The annulus coefficients in direct channel are not all independent. They satisfy a set of polynomial equations:

$$\mathcal{A}_a^{\,i\,b}\,\mathcal{A}_{b\,c}^{\,j} = \sum_k \mathcal{N}^{i\,j}_{\;k}\,\mathcal{A}_{a\,c}^{\,k} \tag{B.2}$$

$$\mathcal{A}_{i\,a\,b}\,\mathcal{A}^{i}{}_{c\,d} = \sum_i \mathcal{A}_{i\,a\,c}\,\mathcal{A}_{b\,d}^{\,i}\,. \tag{B.3}$$

Comments:

- Upper and lower boundary indices in eqs. (B.2) and (B.3) are to be distinguished in presence of oriented boundaries.

- The matrix $(\mathcal{A}_1)_{a\,b} = (\mathcal{A}_1)^{a\,b}$ is a metric for boundary indices, as it follows from eq. (B.2) that

$$\sum_b \mathcal{A}_{i\,a\,b}\,\mathcal{A}_1^{\,b\,c} = \mathcal{A}_{i\,a}^{\,c}\,,$$

  while

$$(\mathcal{A}_1)_a^{\,b} = \delta_a^{\,b}\,. \tag{B.4}$$

- In diagonal models where $\mathcal{A}$ coincides with N, eqs. (B.2) and (B.3) reduce to Verlinde algebra.

- $\mathcal{A}_{i\,a\,b}$ are in general *linearly dependent*, as

$$\sum_i \mathcal{A}_{i\;a\,b}\,S_j^{\,i} = 0\,, \tag{B.5}$$

  where the label j in eq. (B.5) is such a representation for which $Z_{j\,\bar{j}} = 0$ in $\mathcal{T}$.

- *Completeness conditions and reflection coefficients.*

  Define reflection coefficients as:

$$\mathcal{R}_{i\,a} = B_{i\,a}\,\sqrt{S_{i\,0}}\,. \tag{B.6}$$

  Then the completeness conditions(eqs. (B.2) and (B.3)) are satisfied iff [43, 51],

$$\sum_i \mathcal{R}_{i\,a}\,\mathcal{R}^*_{i\,b} = \delta_{a\,b}\,, \qquad \sum_a \mathcal{R}_{i\,a}\,\mathcal{R}^*_{j\,a} = \delta_{i\,j} \tag{B.7}$$

C. Formulae for simple currents

Simple currents were discovered in refs. [52] in order to construct new modular invariants in rational conformal field theory. By definition, a *simple current* is a primary field of the algebra whose fusion with other primary fields produces only one primary field. This is the notion of being *simple*. Thus if J be such a simple current and Φ_i denote a generic primary field of the theory, by definition

$$J \otimes \Phi_i = \Phi_k . \tag{C.1}$$

Every conformal field theory has an obvious simple current, *viz.* the identity. There are examples of non-trivial simple currents in RCFTs. The very existence of a simple current in a RCFT, has some nice consequences; first of all, J will have its conjugate, J^c, such that

$$J \otimes J^c = \mathbf{1} . \tag{C.2}$$

Moreover, products of simple currents are simple currents. Thus powers of J define an *orbit* of simple currents, all of which are different, unless one reaches the identity. Since the number of primary fields in a RCFT is finite, there must exist an integer N, such that

$$J^{\mathsf{N}} = \mathbf{1} . \tag{C.3}$$

If this is the smallest positive integer with such property, then N is called the *order* of the simple current.

The *monodromy charge* associated with a primary field, Φ_i with respect to the simple current J is defined by computing the monodromy of Φ_i around the simple current J. If $Q_J(\Phi_i)$ denote the monodromy charge of the field Φ_i, then

$$J(z)\,\Phi_i(w) \sim (z-w)^{-Q_J(\Phi_i)} \big(J \otimes \Phi_i \big)(w) , \tag{C.4}$$

so that

$$Q_J(\Phi_i) = h_J + h_{\Phi_i} - h_{J \otimes \Phi_i} \mod 1 \tag{C.5}$$

$Q_J(\Phi_i)$ satisfies the following property;

$$Q_J(\Phi_i\, \Phi'_i) = Q_J(\Phi_i) + Q_J(\Phi'_i) \mod 1 . \tag{C.6}$$

The monodromy charge of the simple current is defined as

$$Q_J(J) = \frac{\tilde{r}}{\mathsf{N}} \mod 1 , \tag{C.7}$$

where $\tilde{r}$ is defined modulo N. Using $\Phi_i = J^{n-1}$ in eq. (C.5) and eq. (C.6), one gets a recursion relation for the conformal weights of the currents,

$$h_{J^n} = h_{J^{n-1}} + h_J - (n-1)\, Q_J(J) \mod 1 . \tag{C.8}$$

D. $\mathcal{N} = 2$ minimal models

The k-th $N = 2$ minimal models has central charge $c = 3k/(k+2)$. It can also be realised as the coset $\mathrm{SU}(2)_k \times \mathrm{U}(1)_2 / \mathrm{U}(1)_{k+2}$. The primary fields of the model are specified by two integers (l, m). However, it is useful to split Verma module of a given $\mathcal{N} = 2$ representation into two sectors, those with even or odd worldsheet fermion number. These sectors are distinguished by an extra label, s. Thus, one has three integers (l, m, s) $(l = 0, 1, \ldots, k)$ subject to the constraint that $l + m + s$ is even and the representation $(l, m) = (l, m, s) \oplus (l, m, s+2)$. Even s refers to the NS sector and odd s refers to the R sector fields. The labels have the following field identification given by

$$(l,\, m,\, s) \sim (l,\, m + 2k + 4,\, s) \sim (l,\, m,\, s+4) \sim (k - j,\, m + k + 2,\, s + 2)\,. \tag{D.1}$$

The periodicity conditions in the m and s labels can be fixed by choosing (m, s) values as given below

$$m = -(k+1), -k, \ldots, (k+2)\,, \qquad s = -1, 0, 1, 2\,, \tag{D.2}$$

The dimension h and U(1) charge q of the fields are given by

$$\begin{aligned} h_{l,m,s} &= \Delta_{l,m,s} \mod 1 \\ q_{l,m,s} &= \frac{m}{k+2} - \frac{s}{2} \mod 2 \end{aligned} \tag{D.3}$$

where $\Delta_{l,m,s} \equiv \frac{l(l+2)-m^2}{4(k+2)} + \frac{s^2}{8}$.

The k-th minimal model has a $\mathbb{Z}_{k+2} \times \mathbb{Z}_2$ discrete symmetry. The action of the discrete symmetry on the fields is given by

$$g \cdot \Phi_{l,m,s} = e^{\frac{2\pi i m}{k+2}} \Phi_{l,m,s}\,, \tag{D.4}$$

$$h \cdot \Phi_{l,m,s} = (-)^s \Phi_{l,m,s}\,, \tag{D.5}$$

where g and h generate the $\mathbb{Z}_{k+2}$ and $\mathbb{Z}_2$ respectively.

For our purposes, we need a set of labels in the minimal model that provide a single representative after taking into account all identifications. This is given by the set $\mathsf{FR} = \mathsf{FR}_{\mathrm{NS}} \cup \mathsf{FR}_{\mathrm{R}}$, where

$$\begin{aligned} \mathsf{FR}_{NS} &\equiv \{(l,m,s) \mid 0 \le l \le k\ ;\ |m| \le l\ ;\ s = 0, 2\ ; (l+m) = \text{even}\} \\ \mathsf{FR}_{R} &\equiv \{(l,m,s) \mid 0 \le l \le k\ ;\ |m-1| \le l\ ;\ s = \pm 1\ ; (l+m) = \text{odd}\}\,. \end{aligned} \tag{D.6}$$

We will need the exact conformal weights of these fields. Let $\widetilde{\mathsf{FR}}$ be the set FR without the elements $((0,0,2), (l, l+1, -1))$ $(l = 0, \ldots, k)$. The exact conformal weights are given by

$$h_{l,m,s} = \begin{cases} \Delta_{l,m,s} & \text{for } (l,m,s) \in \widetilde{\mathsf{FR}} \\ \Delta_{l,m,s} + 1 & \text{for } (l,m,s) \in \big((0,0,2), (l, l+1, -1)\big)\ \big(l = 0, \ldots, k\big)\,. \end{cases} \tag{D.7}$$

The crucial part in the computation of the P-matrix is to take the square-root of the T-matrix. We would like to write a formula that is compatible with the various identifications of the (l, m, s). We write it as

$$\big(\sqrt{T}\big)_{l,m,s;l,m,s} = \widehat{\sigma}_{l,m,s} e^{\pi i \Delta_{l,m,s}}\,, \tag{D.8}$$

where $\widehat{\sigma}_{l,m,s}$ is a sign. Recall that $\Delta_{l,m,s}$ gives the weight of the primaries modulo 1. (We defined $h_{l,m,s}$ to be the exact weight.) Consistency with the various identifications, implies that the $\widehat{\sigma}_{l,m,s}$ must satisfy:

$$\widehat{\sigma}_{k-l,m-k-2,s-2} = (-)^{\frac{-l-m+s}{2}} \widehat{\sigma}_{l,m,s}\,,$$

$$\begin{aligned} \widehat{\sigma}_{l,m+2k+4,s} &= (-)^{m+k}\ \widehat{\sigma}_{l,m,s}\,, \\ \widehat{\sigma}_{l,m,s+4} &= (-)^{s}\ \widehat{\sigma}_{l,m,s}\,. \end{aligned} \tag{D.9}$$

In principle, any choice of $\widehat{\sigma}$ that satisfies eq. (D.9) should be acceptable. We however, will choose it such that $\big(\sqrt{T}\big)_{l,m,s;l,m,s} = e^{\pi i h_{l,m,s}}$. This is achieved if we choose:

$$\widehat{\sigma}_{l,m,s} = \begin{cases} 1 & \text{for } (l,m,s) \in \widetilde{\mathsf{FR}} \\ -1 & \text{for } (l,m,s) \in \big((0,0,2),(l,l+1,-1)\big)\ \big(l = 0,\dots,k\big)\,. \end{cases} \tag{D.10}$$

The values of $\widehat{\sigma}$ outside FR is then recursively defined by the relations in eq. (D.9).

D.1 S and P matrices for $\mathrm{U}(1)_k$

The primaries of the $\mathrm{U}(1)_k$ are labelled by an integer m (defined mod $2k$) for which we take the standard range $\mathsf{SR}_k \equiv \{-k+1, -k+2, \dots, k\}$ with weight $h_m = m^2/4k$. The S and P matrices for the $\mathrm{U}(1)_k$ are

$$S_{m\,n} = \frac{1}{\sqrt{2k}} \exp\left[-\frac{\pi i m n}{k}\right] \tag{D.11}$$

$$P_{m\,n} = \frac{1}{\sqrt{k}} \nu_m^{(k)} \nu_n^{(k)} \exp\left[-\frac{\pi i m n}{2k}\right] \delta^{(2)}_{m+n+k} \tag{D.12}$$

where we define $\nu_m^{(k)}$ as follows:

$$\nu_m^{(k)} = \begin{cases} 1 & \text{for } m \in \mathsf{SR}_k \\ (-)^{m+k} & \text{for } (m \pm 2k) \in \mathsf{SR}_k \end{cases} \tag{D.13}$$

with the periodicity $\nu_{m+4k}^{(k)} = \nu_m^{(k)}$. The identity $\nu_{m+2k}^{(k)} = (-)^{m+k} \nu_m^{(k)}$ holds for all m. Note that the addition of the signs given by $\nu_m^{(k)}$ enables us to let values of m go outside the range SR_k.

Using the expressions for the S and P matrices that we gave earlier, one can show that

$$\mathcal{N}^{(k)}_{m_1\,m_2}{}^{m_3} = \delta^{(2k)}_{m_1+m_2-m_3} \tag{D.14}$$

$$Y^{(k)}_{m_1\,m_2}{}^{m_3} = \nu^{(k)}_{m_2} \nu^{(k)}_{m_3}\ \delta^{(2)}_{m_2+m_3} \left[\delta^{(2k)}_{\frac{2m_1+m_2-m_3}{2}} + e^{\pi i(k+m_2)}\ \delta^{(2k)}_{\frac{2m_1+m_2-m_3+2k}{2}}\right]. \tag{D.15}$$

Properties of the Y-tensor: $Y_{(m_1+2k)\,m_2}{}^{m_3} = Y_{m_1\,m_2}{}^{m_3}$

D.2 S and P matrices for $\mathrm{SU}(2)_k$

The S and P matrices for the $\mathrm{SU}(2)_k$ WZW model are

$$\mathcal{S}_{L\,\tilde{L}} = \sqrt{\frac{2}{k+2}}\ \sin\big(L, \tilde{L}\big)_k \tag{D.16}$$

$$\mathcal{P}_{L\,\tilde{L}} = \frac{2}{\sqrt{k+2}}\ \sin\frac{1}{2}\big(L, \tilde{L}\big)_k\ \delta^{(2)}_{L+\tilde{L}+k} \tag{D.17}$$

where $(l, l')_k \equiv (\pi(l+1)(l'+1)/k+2)$.

The $\mathcal{N}$-tensor for this case is

$$\mathcal{N}_{L\,\tilde{L}}{}^{l} = \begin{cases} 1 & |L - \tilde{L}| \le l \le \min\{L+\tilde{L}, 2k - L - \tilde{L}\} \\ 0 & \text{otherwise}\,. \end{cases} \tag{D.18}$$

D.3 Y-tensor elements for $\mathrm{SU}(2)_k$

For convenience we list here the important components of the $\mathrm{SU}(2)_k$ Y-tensor.

$$\begin{aligned} Y_{l\ 0}^{0} &= (-1)^l\,, \qquad Y_{L,k-l}^{k} = \mathcal{N}_{L\ L}^{l}\,, \\ Y_{l\ 0}^{k} &= \mathcal{N}_{l\ k-l}^{0} = \delta_{l,\ k-l}\,. \end{aligned} \tag{D.19}$$

D.4 S and P matrices for the minimal model

Consider the following product of the P-matrices of $\mathrm{SU}(2)_k$, $\mathrm{U}(1)_{k+2}$ and $\mathrm{U}(1)_2$ respectively.

$$\widehat{P}_{LMS\ \tilde{L}\tilde{M}\tilde{S}} \equiv \mathcal{P}_{L\,\tilde{L}} \times P^{*(k+2)}_{M\,\tilde{M}} \times P^{(2)}_{S\,\tilde{S}}\,. \tag{D.20}$$

The S-matrix and the P-matrix of the k-th minimal model is then given by

$$\begin{aligned} S_{LMS\ \tilde{L}\tilde{M}\tilde{S}} &= \frac{1}{\sqrt{2}(k+2)} \sin(l,l')_k \exp\left(\frac{i\pi m m'}{k+2}\right) \exp\left(-\frac{i\pi s s'}{2}\right) \\ P_{LMS\ \tilde{L}\tilde{M}\tilde{S}} &= \frac{1}{2}\widehat{\sigma}_{\tilde{L}\tilde{M}\tilde{S}} \left[\widehat{\sigma}_{LMS}\,\widehat{P}_{LMS\ \tilde{L}\tilde{M}\tilde{S}} + \widehat{\sigma}_{(k-L)(M+k+2)(S+2)}\,\widehat{P}_{(k-L)(M+k+2)(S+2)\ \tilde{L}\tilde{M}\tilde{S}}\right] \end{aligned} \tag{D.21}$$

Using the conditions in eq. (D.9), one can show that the P-matrix is unchanged under all identifications.

D.5 S and P matrices for $\mathrm{SO}(2d)_1$

The four irreps of $\mathrm{SO}(2d)$ are the scalar, vector, spinor and conjugate spinor (O, V, S, C) representations. We will represent them by the label $s_0 = 0, 2, -1, 1$. The S and P matrices are (see the last ref. in [43]) The T matrix is

$$\begin{aligned} T &= e^{-\pi i d/12}\mathrm{diag}\big(1, -1, e^{\pi i d/4}, e^{\pi i d/4}\big) \\ S &= \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & i^{-d} & -i^{-d} \\ 1 & -1 & -i^{-d} & i^{-d} \end{pmatrix}, \qquad P = \begin{pmatrix} c & s & 0 & 0 \\ s & -c & 0 & 0 \\ 0 & 0 & \zeta c & i\zeta s \\ 0 & 0 & i\zeta s & \zeta c \end{pmatrix} \end{aligned} \tag{D.22}$$

where $s = \sin(d\pi/4)$, $c = \cos(d\pi/4)$ and $\zeta = e^{-id\pi/4}$.

D.6 $\mathcal{N} = 2$ characters

For a given representation p of the $\mathcal{N} = 2$ algebra, the character is defined as

$$\chi_p\tau, z, u) = e^{-2i\pi u}\ \mathrm{Tr}_p\left[e^{2i\pi z J_0}\, e^{2i\pi\tau(L_0 - \frac{c}{24})}\right] \tag{D.23}$$

where the trace runs over the particular representation denoted by p and u is an arbitrary phase. The characters of the $N = 2$ minimal models are defined in terms of the level-k theta functions $\Theta_{m,k}(\tau, z, u)$ defined in appendix D.6 and characters of a related parafermionic theory $c^l_m(\tau)$ as:

$$\chi^{(s)}_{l,m}(\tau, z, u) = \sum_{t\in\mathbb{Z}_k} c^l_{m+4t-s}(\tau)\ \Theta_{2m+(4t-s)(k+2),2k(k+2)}\left(\frac{\tau}{2}, \frac{z}{k+2}, u\right). \tag{D.24}$$

The characters $\chi^{(s)}_{l,m}$ have the property that they are invariant under $s \to s+4$ and $m \to m+2(k+2)$ and are zero if $l+m+s \neq 0$ mod 2. For practical purpose, there is another useful formula(equivalent to eq. (D.24)) for the characters [70]

$$\chi^{(s)}_{l,m}\,(\tau, z, u) = \sum_{m'=-k+1}^{k} c^{\,l}_{m'}\,(\tau)\,\Theta_{\,m'(k+2)-mk+2\,s\,k\,,\,k(k+2)}\left(\frac{\tau}{2}\,,\,\frac{z}{k+2}\,,\,u\right) \tag{D.25}$$

$c^l_m(\tau)$ are the famous string functions introduced by Kac and Peterson [73]. We list their symmetry properties below:

$$\begin{aligned} c^l_m &= c^l_{m+2k\mathbb{Z}} = c^l_{-m} = c^{k-l}_{k-m} \\ c^l_m &= 0 \qquad \text{if } l-m \neq 0 \quad \text{mod } 2\,. \end{aligned} \tag{D.26}$$

The level-k theta function is defined as:

$$\begin{aligned} \Theta_{m\,,\,k}(\tau\,,\,z\,,\,u) &= e^{-2\pi i u}\sum_{l\,\in\,Z} e^{2\pi i\tau k\,(l+\frac{m}{2k})^2+2\pi i\,z(l+\frac{m}{2k})} \\ &= e^{-2\pi i u}\sum_{l\,\in\,Z} q^{k\,(l+\frac{m}{2k})^2}\,y^{k\,(l+\frac{m}{2k})} \\ &= e^{-2\pi i u}\,q^{\frac{m^2}{4k}}\,y^{\frac{m}{2}}\,\vartheta_3(kz+m\tau\big|2k\tau)\,, \end{aligned} \tag{D.27}$$

where $q = e^{2\pi i\tau}$ and $y = e^{2\pi i\,z}$ and we have used the following definition of Jacobi's ϑ-functions

$$\vartheta\left[{a \atop b}\right](z|\tau) = \sum_{n\,\in\,\mathbb{Z}} q^{\frac{1}{2}(n+\frac{a}{2})^2}\,e^{2\pi i(z+\frac{b}{2})(n+\frac{a}{2})}\,. \tag{D.28}$$

Following Jacobi/Erdelyi's notation, we have

$$\vartheta_1 = \vartheta\left[{1 \atop 1}\right]\,, \qquad \vartheta_2 = \vartheta\left[{1 \atop 0}\right]\,, \tag{D.29}$$

$$\vartheta_3 = \vartheta\left[{0 \atop 0}\right]\,, \qquad \vartheta_4 = \vartheta\left[{0 \atop 1}\right] \tag{D.30}$$

Level-k theta function has the following symmetry property:

$$\Theta_{m+2k\,,\,k} = \Theta_{m\,,\,k}\,. \tag{D.31}$$

The Weyl-Kac character formula relates level-k SU(2) characters to level- k theta functions through the following identity:

$$\chi^k_l = \frac{\Theta_{l+1\,,\,k+2} - \Theta_{-l-1\,,\,k+2}}{\Theta_{1\,,\,2} - \Theta_{-1\,,\,2}}\,. \tag{D.32}$$

By using the properties of the theta functions, the modular transformation of the minimal model characters is found to be

$$\chi^{(s)}_{l,m}\left(-\frac{1}{\tau},0,0\right) = C\sum_{l',m',s'} \sin(l,l')_k \exp\left(\frac{i\pi mm'}{k+2}\right)\exp\left(-\frac{i\pi ss'}{2}\right)\chi^{(s')}_{l',m'}(\tau,0,0) \tag{D.33}$$

where $(l,l')_k \equiv (\pi(l+1)(l'+1)/k+2)$ and $C = 1/\sqrt{2}(k+2)$.

E. Character formulae of $k = 1$ minimal model

String Functions for $\mathbf{k = 1}$: for $k = 1$, due to its symmetry properties the only non-trivial string functions is $c_0^0(\tau) = c_1^1(\tau)$ [73, 47]. It is actually

$$c_0^0(\tau) = c_1^1(\tau) = \frac{1}{\eta(\tau)}\,. \tag{E.1}$$

Formulae for the NS-sector characters: the supersymmetric characters are defined as:

$$\begin{aligned} A_\pm(2\tau, z) &= (\chi_{00}^0 \pm \chi_{00}^2)(2\tau, z)\,, \qquad B_\pm(2\tau, z) = (\chi_{11}^0 \pm \chi_{11}^2)(2\tau, z) \\ C_\pm(2\tau, z) &= (\chi_{1-1}^0 \pm \chi_{1-1}^2)(2\tau, z)\,. \end{aligned} \tag{E.2}$$

The characaters A_-, B_- and C_- appear in the $(-1)^F$-twisted character $\widetilde{NS}$. The formulae of A_+, B_+ and C_+ interms of theta functions are:

$$A_+(2\tau, z) = \frac{1}{\eta(2\tau)}\,\Theta_{0\,,3}\left(\tau\,,\,\frac{z}{3}\right) = \frac{1}{\eta(2\tau)}\,\vartheta_3(z\big|6\tau)\,, \tag{E.3}$$

$$B_+(2\tau, z) = \frac{1}{\eta(2\tau)}\,\Theta_{2\,,3}\left(\tau\,,\,\frac{z}{3}\right) = q^{\frac{1}{3}}\,y^{\frac{1}{3}}\,\frac{1}{\eta(2\tau)}\,\vartheta_3(z + 2\tau\big|6\tau)\,, \tag{E.4}$$

$$\begin{aligned} C_+(2\tau, z) &= \frac{1}{\eta(2\tau)}\,\Theta_{4\,,3}\left(\tau\,,\,\frac{z}{3}\right) = \frac{1}{\eta(2\tau)}\,\Theta_{-2\,,3}\left(\tau\,,\,\frac{z}{3}\right) \\ &= q^{\frac{1}{3}}\,y^{-\frac{1}{3}}\,\frac{1}{\eta(\tau)}\,\vartheta_3(z - 2\tau\big|6\tau)\,. \end{aligned} \tag{E.5}$$

Formulae for R-sector characters: the supersymmetric characters in R-sectors are:

$$\begin{aligned} \widehat{A}_\pm(2\tau, z) &= (\chi_{01}^1 \pm \chi_{01}^{-1})(2\tau, z)\,, \qquad \widehat{B}_\pm(2\tau, z) = (\chi_{12}^1 \pm \chi_{12}^{-1})(2\tau, z) \\ \widehat{C}_\pm(2\tau, z) &= (\chi_{10}^1 \pm \chi_{10}^{-1})(2\tau, z)\,. \end{aligned} \tag{E.6}$$

The characters $\widehat{A}_-$, $\widehat{B}_-$ and $\widehat{C}_-$ appear in $(-1)^F$-twisted character $\widetilde{R}$.

The R-sector characters are obtained by spectral flowing the NS-sector characters by an amount $\eta \to \eta + \frac{1}{2}$ ($\Rightarrow\ z \to z + \frac{\tau}{2}$) and using the property of $\Theta_{m\,,3}(\tau/2\,,\,z/3)$ under spectral flow:

$$\Theta_{m\,,3}\left(\frac{\tau}{2}\,,\,\frac{z}{3}\right) \xrightarrow{z \to z + \frac{\tau}{2}} q^{-\frac{1}{24}}\,y^{-\frac{1}{6}}\,\Theta_{m+1\,,3}\left(\frac{\tau}{2}\,,\,\frac{z}{3}\right) \xrightarrow{z \to z + \frac{\tau}{2}} q^{-\frac{1}{6}}\,y^{-\frac{1}{3}}\,\Theta_{m+2\,,3}\left(\frac{\tau}{2}\,,\,\frac{z}{3}\right). \tag{E.7}$$

The formulae for $\widehat{A}_+$, $\widehat{B}_+$ and $\widehat{C}_+$ are given below:

$$\widehat{A}_+(2\tau, z) = \frac{1}{\eta(2\tau)}\,\Theta_{1\,,3}\left(\tau\,,\,\frac{z}{3}\right) = \frac{1}{\eta(2\tau)}\,q^{\frac{1}{12}}\,y^{\frac{1}{6}}\,\vartheta_3\big(z + \tau\big|6\tau\big)\,, \tag{E.8}$$

$$\widehat{B}_+(2\tau, z) = \frac{1}{\eta(2\tau)}\,\Theta_{3\,,3}\big(\tau\,,\,\frac{z}{3}\big) = \frac{1}{\eta(2\tau)}\,q^{\frac{3}{4}}\,y^{\frac{1}{2}}\,\vartheta_3\big(z + 3\tau\big|6\tau\big)\,, \tag{E.9}$$

$$\begin{aligned} \widehat{C}_+(2\tau, z) &= \frac{1}{\eta(2\tau)}\,\Theta_{5\,,3}\left(\tau\,,\,\frac{z}{3}\right) = \frac{1}{\eta(2\tau)}\,\Theta_{-1\,,3}\left(\tau\,,\,\frac{z}{3}\right) \\ &= \frac{1}{\eta(2\tau)}\,q^{\frac{1}{12}}\,y^{-\frac{1}{6}}\,\vartheta_3\big(z - \tau\big|6\tau\big)\,. \end{aligned} \tag{E.10}$$

F. Character formulae and spectral flow invariant orbits of $k = 3$ minimal model

String functions: there are four independent string functions at the level $k = 3$. They are $\{c_0^0(\tau)\,,\, c_1^1(\tau)\,,\, c_0^2(\tau)\,,\, c_1^3(\tau)\}$. Their expressions had been given in Kac-Peterson's paper [73]

$$c_1^1(\tau) \;=\; \frac{q^{\frac{3}{40}}\ \prod_{n\neq\pm1 \mod 5}(1-q^{3n})}{\eta(\tau)^2}\,, \tag{F.1}$$

$$c_1^3(\tau) \;=\; \frac{q^{\frac{27}{40}}\ \prod_{n\neq\pm2 \mod 5}(1-q^{3n})}{\eta(\tau)^2}\,, \tag{F.2}$$

$$(c_0^0 - c_1^3)(\tau) \;=\; \frac{q^{\frac{1}{120}}\ \prod_{n\neq\pm1 \mod 5}(1-q^{\frac{n}{3}})}{\eta(\tau)^2}\,, \tag{F.3}$$

$$(c_1^1 - c_0^2)(\tau) \;=\; \frac{q^{\frac{3}{40}}\ \prod_{n\neq\pm2 \mod 5}(1-q^{\frac{n}{3}})}{\eta(\tau)^2} \tag{F.4}$$

Spectral flow: we list the spectral flow of all NS-sector characters in table 5.

Formulae for the characters: the NS-sector characters of $k = 3$ minimal model and their spectral flow are given in the table 5. The formulae of ten characters in NS-sectors

$$\begin{aligned}
A_\pm(2\tau,z) &= (\chi^0_{00} \pm \chi^2_{00})(2\tau,z)\,, & B_\pm(2\tau,z) &= (\chi^0_{1-1} \pm \chi^2_{1-1})(2\tau,z)\\
C_\pm(2\tau,z) &= (\chi^0_{11} \pm \chi^2_{11})(2\tau,z)\,, & D_\pm(2\tau,z) &= (\chi^0_{2-2} \pm \chi^2_{2-2})(2\tau,z)\\
E_\pm(2\tau,z) &= (\chi^0_{20} \pm \chi^2_{20})(2\tau,z)\,, & F_\pm(2\tau,z) &= (\chi^0_{22} \pm \chi^2_{22})(2\tau,z)\\
G_\pm(2\tau,z) &= (\chi^0_{3-3} \pm \chi^2_{3-3})(2\tau,z)\,, & H_\pm(2\tau,z) &= (\chi^0_{3-1} \pm \chi^2_{3-1})(2\tau,z)\\
I_{\pm(2\tau,z)} &= (\chi^0_{31} \pm \chi^2_{31})(2\tau,z)\,, & J_\pm(2\tau,z) &= (\chi^0_{33} \pm \chi^2_{33})(2\tau,z)\,.
\end{aligned} \tag{F.5}$$

The characters A_+, B_+, C_+, D_+, E_+, F_+, G_+, H_+, I_+ and J_+ can be obtained from the following level-15 theta function, by changing the values of m:

$$\Theta_{m\,,15}\left(\tau, \frac{z}{5}, 0\right) = q^{m^2/60}\ y^{m/10}\ \vartheta_3\big(3z + m\tau\big|30\tau\big)\,. \tag{F.6}$$

The R-sector characters are

$$\begin{aligned}
\widehat{A}_\pm(2\tau,z) &= (\chi^1_{01} \pm \chi^{-1}_{01})(2\tau,z)\,, & \widehat{B}_\pm(2\tau,z) &= (\chi^1_{10} \pm \chi^{-1}_{10})(2\tau,z)\\
\widehat{C}_\pm(2\tau,z) &= (\chi^1_{12} \pm \chi^{-1}_{12})(2\tau,z)\,, & \widehat{D}_\pm(2\tau,z) &= (\chi^1_{2-1} \pm \chi^{-1}_{2-1})(2\tau,z)\\
\widehat{E}_\pm(2\tau,z) &= (\chi^1_{21} \pm \chi^{-1}_{21})(2\tau,z)\,, & \widehat{F}_\pm(2\tau,z) &= (\chi^1_{23} \pm \chi^{-1}_{23})(2\tau,z)\\
\widehat{G}_\pm(2\tau,z) &= (\chi^1_{3-2} \pm \chi^{-1}_{3-2})(2\tau,z)\,, & \widehat{H}_\pm(2\tau,z) &= (\chi^1_{30} \pm \chi^{-1}_{30})(2\tau,z)\\
\widehat{I}_{\pm(2\tau,z)} &= (\chi^1_{32} \pm \chi^{-1}_{32})(2\tau,z)\,, & \widehat{J}_\pm(2\tau,z) &= (\chi^1_{34} \pm \chi^{-1}_{34})(2\tau,z)\,.
\end{aligned} \tag{F.7}$$

References

[1] B.S. Acharya, *M-theory, Joyce orbifolds and super Yang-Mills*, *Adv. Theor. Math. Phys.* **3** (1999) 227 [hep-th/9812205].

[2] B.S. Acharya, *On realising $N = 1$ super Yang-Mills in M-theory*, hep-th/0011089.

[3] B.S. Acharya, *Confining strings from G_2-holonomy spacetimes*, hep-th/0101206.

[4] B.S. Acharya and C. Vafa, *On domain walls of $N = 1$ supersymmetric Yang-Mills in four dimensions*, hep-th/0103011.

[5] M. Atiyah and E. Witten, *M-theory dynamics on a manifold of G_2 holonomy*, *Adv. Theor. Math. Phys.* **6** (2003) 1 [hep-th/0107177].

[6] E. Witten, *Anomaly cancellation on manifolds of G_2 holonomy*, hep-th/0108165.

[7] B. Acharya and E. Witten, *Chiral fermions from manifolds of G_2 holonomy*, hep-th/0109152.

[8] T. Friedmann, *On the quantum moduli space of M-theory compactifications*, *Nucl. Phys.* **B 635** (2002) 384 [hep-th/0203256].

[9] A. Bilal and S. Metzger, *Compact weak G_2-manifolds with conical singularities*, *Nucl. Phys.* **B 663** (2003) 343 [hep-th/0302021].

[10] B.S. Acharya, *M-theory, G_2-manifolds and four-dimensional physics*, *Class. and Quant. Grav.* **19** (2002) 5619.

[11] D.D. Joyce, *Compact riemannian 7-manifolds with holonomy G_2, I*, *J. Diff. Geom.* **43** (1996) 291.

[12] D.D. Joyce, *Compact riemannian 7-manifolds with holonomy G_2, II*, *J. Diff. Geom.* **43** (1996) 329.

[13] S. Kachru and J. McGreevy, *M-theory on manifolds of G_2 holonomy and type-IIA orientifolds*, *J. High Energy Phys.* **06** (2001) 027 [hep-th/0103223].

[14] M. Cvetič, G. Shiu and A.M. Uranga, *Three-family supersymmetric standard like models from intersecting brane worlds*, *Phys. Rev. Lett.* **87** (2001) 201801 [hep-th/0107143].

[15] M. Cvetič, G. Shiu and A.M. Uranga, *Chiral four-dimensional $N = 1$ supersymmetric type-IIA orientifolds from intersecting d6-branes*, *Nucl. Phys.* **B 615** (2001) 3 [hep-th/0107166].

[16] M. Cvetič, G. Shiu and A.M. Uranga, *Chiral type-II orientifold constructions as M-theory on G_2 holonomy spaces*, hep-th/0111179.

[17] A. Sagnotti, *Open strings and their symmetry groups*, hep-th/0208020.

[18] C.G. Callan Jr., C. Lovelace, C.R. Nappi and S.A. Yost, *Adding holes and crosscaps to the superstring*, *Nucl. Phys.* **B 293** (1987) 83; *Loop corrections to superstring equations of motion*, *Nucl. Phys.* **B 308** (1988) 221; *Loop corrections to conformal invariance for type 1 superstrings*, *Phys. Lett.* **B 206** (1988) 41.

[19] J. Polchinski and Y. Cai, *Consistency of open superstring theories*, *Nucl. Phys.* **B 296** (1988) 91.

[20] P. Hořava, *Background duality of open string models*, *Phys. Lett.* **B 231** (1989) 251; *Strings on world sheet orbifolds*, *Nucl. Phys.* **B 327** (1989) 461.

[21] B. Acharya, M. Aganagic, K. Hori and C. Vafa, *Orientifolds, mirror symmetry and superpotentials*, hep-th/0202208.

[22] R. Blumenhagen, V. Braun, B. Körs and D. Lüst, *Orientifolds of K3 and Calabi-Yau manifolds with intersecting D-branes*, *J. High Energy Phys.* **07** (2002) 026 [hep-th/0206038].

[23] A. Misra, *On (orientifold of) type-IIA on a compact Calabi-Yau*, *Fortschr. Phys.* **52** (2004) 5 [hep-th/0304209].

[24] D.-E. Diaconescu, B. Florea and A. Misra, *Orientifolds, unoriented instantons and localization*, *J. High Energy Phys.* **07** (2003) 041 [hep-th/0305021].

[25] M. Bianchi and A. Sagnotti, *On the systematics of open string theories*, *Phys. Lett.* **B 247** (1990) 517; *Twist symmetry and open string wilson lines*, *Nucl. Phys.* **B 361** (1991) 519; M. Bianchi, G. Pradisi and A. Sagnotti, *Toroidal compactification and symmetry breaking in open string theories*, *Nucl. Phys.* **B 376** (1992) 365.

[26] I. Brunner, *On orientifolds of WZW models and their relation to geometry*, *J. High Energy Phys.* **01** (2002) 007 [hep-th/0110219].

[27] Y. Hikida, *Orientifolds of* $\mathrm{SU}(2)/\mathrm{U}(1)$ *WZW models*, *J. High Energy Phys.* **11** (2002) 035 [hep-th/0201175]; *Crosscap states for orientifolds of euclidean* AdS_3, *J. High Energy Phys.* **05** (2002) 021 [hep-th/0203030].

[28] L.R. Huiszoon, K. Schalm and A.N. Schellekens, *Geometry of WZW orientifolds*, *Nucl. Phys.* **B 624** (2002) 219 [hep-th/0110267].

[29] M.B. Halpern and C. Helfgott, *Twisted open strings from closed strings: the WZW orientation orbifolds*, hep-th/0306014.

[30] J. Fuchs, I. Runkel and C. Schweigert, *Tft construction of rCFT correlators. II: Unoriented world sheets*, *Nucl. Phys.* **B 678** (2004) 511 [hep-th/0306164].

[31] I. Brunner and K. Hori, *Notes on orientifolds of rational conformal field theories*, hep-th/0208141.

[32] I. Brunner and K. Hori, *Orientifolds and mirror symmetry*, hep-th/0303135.

[33] R. Blumenhagen and A. Wisskirchen, *Spectra of 4D,* $N = 1$ *type-I string vacua on non-toroidal CY threefolds*, *Phys. Lett.* **B 438** (1998) 52 [hep-th/9806131].

[34] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y.S. Stanev, *Comments on gepner models and type-I vacua in string theory*, *Phys. Lett.* **B 387** (1996) 743 [hep-th/9607229].

[35] J.M. Maldacena, G.W. Moore and N. Seiberg, *Geometrical interpretation of D-branes in gauged WZW models*, *J. High Energy Phys.* **07** (2001) 046 [hep-th/0105038].

[36] A. Recknagel and V. Schomerus, *D-branes in Gepner models*, *Nucl. Phys.* **B 531** (1998) 185 [hep-th/9712186].

[37] I. Brunner, M.R. Douglas, A.E. Lawrence and C. Romelsberger, *D-branes on the quintic*, *J. High Energy Phys.* **08** (2000) 015 [hep-th/9906200].

[38] S. Govindarajan, T. Jayaraman and T. Sarkar, *Worldsheet approaches to D-branes on supersymmetric cycles*, *Nucl. Phys.* **B 580** (2000) 519 [hep-th/9907131]; *On D-branes from gauged linear sigma models*, *Nucl. Phys.* **B 593** (2001) 155 [hep-th/0007075]; S. Govindarajan and T. Jayaraman, *Boundary fermions, coherent sheaves and D-branes on Calabi-Yau manifolds*, *Nucl. Phys.* **B 618** (2001) 50 [hep-th/0104126].

JHEP02(2004)026

[39] A. Sen, *A note on enhanced gauge symmetries in M- and string theory*, *J. High Energy Phys.* **09** (1997) 001 [hep-th/9707123].

[40] J. Majumder, *Type-IIA orientifold limit of M-theory on compact Joyce 8-manifold of* Spin(7)*-holonomy*, *J. High Energy Phys.* **01** (2002) 048 [hep-th/0109076].

[41] C. Vafa and E. Witten, *Dual string pairs with $N = 1$ and $N = 2$ supersymmetry in four dimensions*, *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 225 [hep-th/9507050].

[42] J.L. Cardy, *Boundary conditions, fusion rules and verlinde formula*, *Nucl. Phys.* **B 324** (1989) 581.

[43] G. Pradisi, A. Sagnotti and Y.S. Stanev, *Planar duality in* SU(2) *WZW models*, *Phys. Lett.* **B 354** (1995) 279 [hep-th/9503207]; *The open descendants of nondiagonal* SU(2) *WZW models*, *Phys. Lett.* **B 356** (1995) 230 [hep-th/9506014]; *Completeness conditions for boundary operators in 2d conformal field theory*, *Phys. Lett.* **B 381** (1996) 97 [hep-th/9603097];
A. Sagnotti and Y.S. Stanev, *Open descendants in conformal field theory*, *Fortschr. Phys.* **44** (1996) 585 [hep-th/9605042];
P. Bantay, *The Frobenius-Schur indicator in conformal field theory*, *Phys. Lett.* **B 394** (1997) 87 [hep-th/9610192]; *Frobenius-Schur indicators, the Klein-bottle amplitude and the principle of orbifold covariance*, *Phys. Lett.* **B 488** (2000) 207 [hep-th/0001173];
Y.S. Stanev, *Two dimensional conformal field theory on open and unoriented surfaces*, hep-th/0112222;
C. Angelantonj and A. Sagnotti, *Open strings*, *Phys. Rept.* **371** (2002) 1 [hep-th/0204089].

[44] T. Eguchi, H. Ooguri, A. Taormina and S.-K. Yang, *Superconformal algebras and string compactification on manifolds with* SU(N) *holonomy*, *Nucl. Phys.* **B 315** (1989) 193.

[45] S. Odake, *Extension of $N = 2$ superconformal algebra and Calabi-Yau compactification*, *Mod. Phys. Lett.* **A 4** (1989) 557; *Character formulas of an extended superconformal algebra relevant to string compactification*, *Int. J. Mod. Phys.* **A 5** (1990) 897.

[46] M. Berkooz, M.R. Douglas and R.G. Leigh, *Branes intersecting at angles*, *Nucl. Phys.* **B 480** (1996) 265 [hep-th/9606139].

[47] D. Gepner, *Spacetime supersymmetry in compactified string theory and superconformal models*, *Nucl. Phys.* **B 296** (1987) 757; *Lectures on $N = 2$ theory*, published in *Superstrings '89*, M. Green et al. eds., Trieste, Italy, Apr 3-14, 1989.

[48] S. Govindarajan and T. Jayaraman, *On the Landau-Ginzburg description of boundary cfts and special lagrangian submanifolds*, *J. High Energy Phys.* **07** (2000) 016 [hep-th/0003242].

[49] E. Verlinde , *Fusion rules and modular transformations in 2-D conformal field theory*, *Nucl. Phys.* **B 300** (1988) 360.

[50] G. Pradisi and A. Sagnotti, *Open string orbifolds*, *Phys. Lett.* **B 216** (1989) 59.

[51] R.E. Behrend, P.A. Pearce, V.B. Petkova and J.-B. Zuber, *On the classification of bulk and boundary conformal field theories*, *Phys. Lett.* **B 444** (1998) 163 [hep-th/9809097]; *Boundary conditions in rational conformal field theories*, *Nucl. Phys.* **B 570** (2000) 525 [hep-th/9908036].

[52] A.N. Schellekens and S. Yankielowicz, *Extended chiral algebras and modular invariants partition functio*, *Nucl. Phys.* **B 327** (1989) 673;
K. Intriligator, *Bonus symmetry in conformal field theory*, npb3321990541;

A.N. Schellekens and S. Yankielowicz, *Modular invariants from simple currents: an explicit proof*, *Phys. Lett.* **B 227** (1989) 387; *Simple currents, modular invariants and fixed points*, *Int. J. Mod. Phys.* **A 5** (1990) 2903.

[53] L.R. Huiszoon, A.N. Schellekens and N. Sousa, *Klein bottles and simple currents*, *Phys. Lett.* **B 470** (1999) 95 [hep-th/9909114];
J. Fuchs, L.R. Huiszoon, A.N. Schellekens, C. Schweigert and J. Walcher, *Boundaries, crosscaps and simple currents*, *Phys. Lett.* **B 495** (2000) 427 [hep-th/0007174].

[54] N. Ishibashi, *The boundary and crosscap states in conformal field theories*, *Mod. Phys. Lett.* **A 4** (1989) 251.

[55] T. Onogi and N. Ishibashi, *Conformal field theories on surfaces with boundaries and crosscaps*, *Mod. Phys. Lett.* **A 4** (1989) 161.

[56] A. Sen, *Non-BPS states and branes in string theory*, hep-th/9904207.

[57] G. Papadopoulos and P.K. Townsend, *Compactification of $D = 11$ supergravity on spaces of exceptional holonomy*, *Phys. Lett.* **B 357** (1995) 300 [hep-th/9506150].

[58] J.A. Harvey and G.W. Moore, *Superpotentials and membrane instantons*, hep-th/9907026.

[59] H. Partouche and B. Pioline, *Rolling among G_2 vacua*, *J. High Energy Phys.* **03** (2001) 005 [hep-th/0011130].

[60] R. Roiban, C. Romelsberger and J. Walcher, *Discrete torsion in singular G_2-manifolds and real LG*, *Adv. Theor. Math. Phys.* **6** (2003) 207 [hep-th/0203272].

[61] B.S. Acharya, *A moduli fixing mechanism in M-theory*, hep-th/0212294.

[62] J. Fuchs, A. Klemm, C. Scheich and M.G. Schmidt, *Spectra and symmetries of gepner models compared to Calabi-Yau compactifications*, *Ann. Phys. (NY)* **204** (1990) 1.

[63] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, *Phys. Rept.* **244** (1994) 77 [hep-th/9401139].

[64] K. Becker, M. Becker and A. Strominger, *Five-branes, membranes and nonperturbative string theory*, *Nucl. Phys.* **B 456** (1995) 130 [hep-th/9507158].

[65] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality*, *Nucl. Phys.* **B 479** (1996) 243 [hep-th/9606040].

[66] N.P. Warner, *N=2 supersymmetric integrable models and topological field theories*, hep-th/9301088.

[67] B.R. Greene, *String theory on Calabi-Yau manifolds*, hep-th/9702155.

[68] R. Blumenhagen and V. Braun, *Superconformal field theories for compact G_2 manifolds*, *J. High Energy Phys.* **12** (2001) 006 [hep-th/0110232].

[69] S. Cecotti, S. Ferrara and L. Girardello, *Geometry of type-II superstrings and the moduli of superconformal field theories*, *Int. J. Mod. Phys.* **A 4** (1989) 2475.

[70] F. Ravanini and S.-K. Yang, *Modular invariance in $N = 2$ superconformal field theories*, *Phys. Lett.* **B 195** (1987) 202.

[71] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton 1992.

[72] E.G. Gimon and J. Polchinski, *Consistency conditions for orientifolds and d-manifolds*, *Phys. Rev.* **D 54** (1996) 1667 [hep-th/9601038].

[73] V.G. Kac and D.H. Peterson, *Infinite Lie algebras, theta functions and modular forms*, *Adv. Math.* **53** (1984) 125.

[74] S. Govindarajan and J. Majumder, work in progress.

[75] M.R. Douglas, B. Fiol and C. Romelsberger, *Stability and BPS branes*, hep-th/0002037.

[76] B.R. Greene and M.R. Plesser, *Duality in Calabi-Yau moduli space*, *Nucl. Phys.* **B 338** (1990) 15.

[77] S. Govindarajan, *Orientifolds of type-IIA strings on Calabi-Yau manifolds*, in *Online proceedings at IXth international symposium on particles, strings and cosmology, PASCOS 2003* held at Tata Institute of Fundamental Research, Mumbai, India, Januray 3–8, 2003, http://malkauns.theory.tifr.res.in/proceedings/sunday/p3/suresh/index.html, [hep-th/0305108].

[78] L.R. Huiszoon, K. Schalm, *BPS orientifold planes from crosscap states in Calabi-Yau compactifications*, hep-th/030609

[79] S. Govindarajan and J. Majumder, *Orientifolds of type-IIA strings on Calabi-Yau manifolds*, hep-th/0305108.

[80] S. Govindarajan and T. Jayaraman, to appear.